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RELIABLE DECENTRALIZED CONTROLLER DESIGN FOR DISCRETE/CONTINUOUS-TIME, SAMPLED-DATA, AND MULTIRATE SYSTEMS

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RELIABLE DECENTRALIZED CONTROLLER DESIGN FOR DISCRETE/CONTINUOUS-TIME, SAMPLED-DATA, AND MULTIRATE SYSTEMS

BY

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A.B., Harvard University, 1984
M.S., University of Illinois, 1987

THESIS

Submitted in partial fulfillment of the requirements
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in the Graduate College of the
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RELIABLE DECENTRALIZED CONTROLLER DESIGN
FOR DISCRETE/CONTINUOUS-TIME, SAMPLED-DATA, AND MULTIRATE SYSTEMS

Molly Hathaway Shor
Department of Electrical and Computer Engineering
University of Illinois at Urbana-Champaign, 1992
W. R. Perkins, Advisor

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DEDICATION

To two strong women

Joan Bopp Shor and Dorothy Williston Shor

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CHAPTER 1

INTRODUCTION

A decentralized controller structure may be the natural one for a large dynamical system. First, maintaining constant communication among all parts of a large system may be difficult or costly. Thus, only a subset of the sensor outputs may be readily available at each actuator input, making a decentralized controller structure the natural choice. Second, a decentralized design may permit the enhancement of overall system reliability. In a centralized controller structure, loss of the lone microprocessor is generally catastrophic. Redundancy of processors may permit reliability to controller outages if proper consideration is given in the design to reliability. This thesis deals specifically with reliability to controller sensor and actuator channel outages for both centralized and decentralized controller structures. For the decentralized controller structure, loss of a controller processor would be modelled as the loss of the controller actuator channel associated with the processor if the processor failure resulted in a zero output.

Controllers are generally designed for the nominal system, the system as modelled with all components functional. Robust designs have been developed that guarantee performance for both the nominal system and the system with a "small" amount of unmodelled dynamics, as measured by some norm. Sensor or actuator outages, however, create large structural perturbations in the system dynamical model, not small ones. A controller that has been designed for the nominal system, or for the nominal system with a small amount of unmodelled dynamics, may lead to catastrophic performance such as instability in the system with outages in the sensors and actuators.

In designs that must be reliable to sensor and actuator outages, individual controller designs are generally found for each possible outage, and a fault detection and reconfiguration system is employed to switch among controller designs when an outage is detected. Detection and reconfiguration may be expensive, and in some cases it may be more difficult to detect on-line the exact failure that has occurred than to design a single controller that guarantees performance for an entire set of possible failures.

In this thesis, observer-based controller designs are presented that are robust to failures from a preselected set of sensors or actuators for both centralized and decentralized systems. The designs guarantee stability and an H_∞-norm bound on the closed-loop system despite failures in any subset of that preselected set of sensors or actuators.

The discrete-time divided-difference operator formulation facilitates the solution of three problems: the unification of discrete- and continuous-time results for sampled-data systems, providing a numerical method for solving the decentralized discrete-time design equation; the design of sampled-data decentralized controllers that bound the *continuous-time* closed-loop system norm; and the design of multirate decentralized digital controllers for systems with sensors and actuators operating at different sampling and zero-order-hold rates.

This chapter begins with a discussion on the "reliable" control problem, a description of the problems solved in this thesis, and a summary of background results and contributions of this thesis in each of those problem areas. An introduction to the divided-difference operator formulation for sampled-data systems follows, and the chapter concludes with an outline of the body of the thesis.

1.1 Reliable Control: Robustness to Sensor and Actuator Failure

Until recently, fault recovery depended entirely on the detection of system failures and reconfiguration of the system, or on the duplication of system components or subsystems and a voting scheme to determine which component or subsystem was operational and should be used. The overhead for each of these schemes is relatively high. Furthermore, they may not efficiently take advantage of redundancy in non-co-located sensors and actuators.

Additional sensors and actuators in a system may be redundant, even when they are not co-located, in the sense that the system may perform satisfactorily with only a subset of them functioning. Whether immediate detection of a particular fault and reconfiguration are required depends on whether a single controller may be designed that will function satisfactorily for the system while both fully functional and in the presence of that fault. This problem has been addressed recently for prespecified sets of sensor or actuator outages by Veillette, Paz, Medanić and Perkins in their development of "reliable" controller designs [1], [2], [3].

Since the term "reliable" is already used to refer to those schemes involving highly redundant component and controller placement, these new controller designs might more accurately be described as "fault-tolerant" since they produce controllers that are insensitive to certain faults occurring in the system. These controllers could also be described as robust to sensor and actuator failures.

More specifically, Veillette, Paz, Medanić and Perkins have developed observer-based controller designs that guarantee stability and an H_∞-norm bound on the closed-loop system despite outage failures in any subset of a preselected set of sensors or actuators. The continuous-time reliable, or fault-tolerant, controller designs for both centralized and decentralized controllers were presented by Veillette, Medanić and Perkins in [1]. Discrete-time reliable controller designs were given by Paz and Medanić in [2], [3]. Paz and Medanić, however, encountered numerical difficulties in the solution of the decentralized design equations.

Other "reliable" controller designs of this type have been considered for decentralized-system interconnections not involving sensors and actuators by Ünyelioğlu and Özgüler in [4]. They consider controllers that stabilize the closed-loop system despite outages of feedforward and feedback interconnections.

1.2 Problems Solved in This Thesis

In this thesis, reliable controller designs, in the sense of Veillette, Paz, Medanić and Perkins, are developed for three new problem setups.

There are, in fact, two control design problems that are solved in each case: the basic H_∞-norm-bounding control problem and the reliable control problem. For the H_∞-norm-bounding control problem, a controller is designed that guarantees that the closed-loop system is stable and that the H_∞ norm of the closed-loop system, measured from the vector of all disturbances and noises entering the system to the regulated-output variable, is bounded by a prespecified value α.

For the reliable control problem, a controller is designed that guarantees both closed-loop stability and the closed-loop H_∞-norm bound for both the original system and for the system with a certain prespecified class of sensor, or actuator, outages. The outages are restricted to be a subset, any subset, of a prespecified set of sensors, or actuators. Further, sensors, or actuators, experiencing an outage are assumed to have output signal zero.

In Chapter 2, the discrete- and continuous-time reliable controller designs of Veillette, Paz, Medanić and Perkins are unified using the techniques advocated by Middleton and Goodwin in [5]. Both centralized and decentralized observer-based controller designs are given that solve the H_∞-norm-bounding and reliable control problems. Both sensor- and actuator-outage reliable problems are treated. In addition, a unified notion of degree of stability for continuous-time and sampled-data systems is presented, and a decentralized H_∞-norm-bounding controller design is given that provides a prescribed degree of stability for the closed-loop system. Finally, a new numerical method is presented that permits the solution of the discrete-time decentralized design equation.

In Chapter 3, reliable controller designs are developed for sampled-data decentralized controllers that explicitly bound the *continuous-time* closed-loop system norm. State-feedback and decentralized controller designs are obtained that solve the H_∞-norm-bounding problem, and decentralized controller designs are developed that solve the sensor-outage reliable control problem. Explicit design equations are obtained in each case. A new approach of intrinsic interest is used to develop these sampled-data controller designs.

In Chapter 4, H_∞-norm-bounding and sensor-outage reliable decentralized controller designs are developed for multirate digital systems with sensors and actuators operating at different sampling and zero-order-hold rates. A simplifying assumption is required for the reliable controller design, resulting in a slightly more conservative bound on the controller. The norm for the multirate system is selected to correspond to the underlying continuous-time norm so that the design norm need not be redesigned for the multirate system.

1.3 Background and Methodology

In this section, the divided-difference operator formulation for sampled-data systems, the problem of designing sampled-data controllers that explicitly bound the continuous-time H_∞ norm, the lifting technique used to solve multirate control problems, and the design methodology for H_∞-norm-bounding and reliable controller design are discussed. Prior results in these areas are discussed, and new developments in the approach taken are noted. The section concludes with a discussion of the solution method used in this thesis to obtain H_∞-norm-bounding and reliable controller designs.

1.3.1 Divided-difference operator formulation

Recently, Middleton and Goodwin formalized in [5] the divided-difference operator and unified approaches for dealing with continuous-time linear systems and their sampled-data zero-order-hold equivalents.

Previously, sampled-data dynamical systems had been written as general discrete-time dynamical systems in difference-equation form, with the state at the next sampling instance expressed in terms of the state at the current sampling instance. The frequency-domain analogue of the difference-equation form was the z-transform description. One difficulty with the use of this form for implementation purposes is that system dynamics information is lost in the round-off errors for fast sampling, resulting in loss of numerical precision. This occurs because the state transition matrix approaches the identity matrix I as the sampling interval T tends to zero. While numerical methods have been developed for computing discrete-time Riccati equation solutions, et cetera, Goodwin, Middleton, Poor, Vijayan, Moore, and others have shown that the use of the divided-difference operator approach permits improved numerical results [5], [6], [7]. Another difficulty with the difference-equation form for sampled-data systems is that the relationship between the sampled-data system and the underlying continuous-time system is obscured. The divided-difference formulation makes this relationship more explicit, which can prove useful.

The divided-difference formulation is the treatment of sampled-data dynamical systems as they are treated in an introduction to calculus. The state equation is written as the difference between the state at successive sampling instances, normalized by the time between sampling instances, called the sampling interval. In the limit as the sampling interval approaches zero, this normalized difference approaches the derivative of the state variable. Similarly, summations are rewritten, for regular sampling, as the sampling interval multiplied by the summation, which is a particular Riemann sum. If all of the variables in the summation are Riemann integrable, this converges to an integral as the sampling interval approaches zero.

Since the matrices and variables in the divided-difference formulation are close, and converge, to the corresponding continuous-time matrices and variables, the information losses experienced when implementing the difference-equation formulation for fast sampling are avoided. This property is also utilized in Chapter 2 to develop an iterative numerical scheme for solving the decentralized design equation by starting the iteration at the continuous-time solution.

The Riemann sums in the divided-difference formulation can be seen from a second viewpoint. They are precisely the integral of a piecewise-constant variable. Thus, they are exact integrals of signals that are the outputs of zero-order holds. This is useful in the development of the norms for the sample-data and multirate cases.

Previous publications on the divided-difference formulation relating directly to the results in this thesis include the work of Lee, Middleton, Goodwin, and Kolodziej, who published the solution to the state-feedback H_∞-optimal control problem in the unified form and related it to the dynamic game problem [5], [8].

1.3.2 Sampled-data controller design: continuous-time norm problems

Traditionally, digital controllers have been designed for continuous-time plants either by designing an appropriate continuous-time controller and discretizing it, or by discretizing the plant and designing a discrete-time controller for the resulting discrete-time system. The first approach neglects the resulting errors in the implementation, which may compound over time. The second approach guarantees performance at the sampling times but disregards performance between sampling times.

It is generally supposed that, if sampling is "fast enough," either of these approaches will result in satisfactory performance. However, fast sampling is not always convenient since many actuators and sensors have limited time-response characteristics.

Recent breakthroughs yield sampled-data controller designs that optimize closed-loop system behavior over all time, not just at sampling instances. This has sparked renewed interest in the problem and prompted a number of solutions to the basic problem. A new approach is presented in this thesis that provides more insight into the development of these new design methods as well as demonstrating which signal norm should be considered for the mixed continuous and piecewise-constant signal space.

Başar [9] derives the optimal time-varying controller design that minimizes the H_∞ norm of a continuous-time plant with sampled measurements, using a game-theoretic approach. Toivonen treats the H_∞-optimal finite-horizon control problem for time-varying continuous-time systems with sampled-data controllers, piecewise constant on each sampling interval, using a game-theoretic approach, resulting in a series of Riccati equations that must be solved in reverse time [10], or in a finite time-horizon discrete-time filtering problem of a different form from that of the original problem [11].

Bamieh and Pearson [12] and Toivonen [11] also treat the H_∞-optimal control problem for continuous-time systems with sampled-data controllers, piecewise constant on each sampling interval, in the frequency domain by using a lifting of the continuous-time signals into an infinite-dimensional signal space and then by showing that the part of the signal of interest is actually finite-dimensional. Their infinite-dimensional problem then reduces to a finite-dimensional discrete-time H_∞ problem of a different form from the original problem.

In Chapter 3, H_∞-norm-bounding state-feedback and decentralized sampled-data controller designs, and reliable decentralized sampled-data controller designs, piecewise constant on each control sampling interval, are found as the limit of a convergent sequence of decentralized two-rate controller designs as the sampling rate on the disturbance and regulated-output variables tends to infinity. The sampling rate on the disturbance and regulated-output variables is chosen to be a multiple of the zero-order-hold rate on the control variable so that performance is measured, in the limit, over all time. The norm for the two-rate problems is chosen to converge in the limit to the continuous-time norm using an approximation of the continuous-time norm as the quotient of Riemann sums. Design equations are then derived for the two-rate controller, and the limiting controller is found as the sampling interval on the disturbance and regulated-output variables tends to zero.

1.3.3 Multirate controller design

Systems often need to be controlled in practice using sensors and actuators with different time constants. If the slowest sensor or actuator determines the sampling interval of the system, certain performance goals may not be attainable. As a result, sensors and actuators are permitted to function at different, rationally related, rates, resulting in a multirate sampled-data system.

A lifting of the multirate digital system to a single-rate equivalent digital system was proposed by Meyer and Burrus [13] and generalized by Buescher and Grizzle [14], [15], and D. G. Meyer [16]. A problem encountered when using the lifted system for controller design was that the norm of the lifted system did not correlate with the norm of the underlying continuous-time system. Recently, Al-Rahmani and Franklin proposed a new lifting for multirate sampled-data systems [17]. This new lifting is not considered in this thesis but encounters the same problem in the choice of an appropriate norm for the lifted system.

In Chapter 4, controller designs are developed for multirate decentralized digital controllers for systems with sensors and actuators operating at different sampling and zero-order-hold rates. The norm for the multirate system is chosen to correspond to the underlying continuous-time norm in the same sense as in Chapter 3. The resulting multirate problem is then solved using a lifting to a single-rate problem. The advantage is that a good design norm need not be redesigned for the multirate problem. If a suitable design norm is selected for the continuous-time system, the corresponding norm for use with the multirate system follows.

One cautionary note is raised, however, on the use of Riemann sums as approximations to integrals when not taking the limit. Chen and Francis [18] point out that the sequence generated from sampling a continuous $\mathcal{L}_2$ signal is not necessarily in ℓ_2. Thus, for any fixed sampling interval T, the Riemann sum may not be close to the integral for every signal in $\mathcal{L}_2$.

1.3.4 Solution method to H_∞-norm-bounding and reliable control problems

The basic approach to the solution of the H_∞-norm-bounding and reliable controller design problems is the application of a bounded real lemma, which was adapted by Veillette for the continuous-time case from a result of Willems [19]. The bounded real lemma in the unified formulation for the cases discussed in Chapter 2 combines the lemmas of Veillette and Paz, adapting the basic approach of Paz's proof to the divided-difference/unified formulation. A modification to the lemma provides a prescribed degree of stability for the system, and a more general bounded real lemma is proved in Chapter 3 to handle the throughput disturbance terms in the regulated-output equation that appear in a lifted "two-rate" system, which arises if the disturbance and regulated-output variables are sampled at a faster rate than the measured-output variable and the zero-order-hold rate of the control variable. It is applied again in Chapter 4 to find controller designs for multirate control systems.

The form of the closed-loop system is found for selected forms of the controller, and conditions are found on the controller gains to guarantee that the hypotheses of the bounded real lemma hold. The consequence of the lemma is that the closed-loop system is stable and has the desired H_∞-norm bound. For the reliable control problems, conditions are found that guarantee that the hypotheses of the bounded real lemma hold for all combinations of failures in the prespecified set.

In Chapters 3 and 4, a single-rate form of the discrete-time system is first obtained, with the appropriate regulated-output variable derived by finding a form of the H_∞ norm corresponding to the H_∞ norm of the underlying continuous-time system. Then the controller form is selected to guarantee causality of the controller, and the bounded real lemma's hypotheses are considered, as before.

Other design methodologies have been used to design H_∞-norm-bounding centralized and decentralized controllers. For instance, the optimal (H_∞-norm-minimizing) stabilizing state-feedback and centralized output-feedback controllers have been found by posing the problems as dynamic game problems. References to this work may be found in [20]. The optimal decentralized controllers have also been found by Didinsky and Başar by posing the problems as dynamic game problems [21], [22]. The resulting conditions for the state-feedback case have been presented in the unified formulation in [5] and [8]. However, the methodology of this thesis, and of Veillette, Paz, Medanić and Perkins's publications [1], [3], [2], permits the achievement of more general design goals including reliability to sensor and actuator failures.

1.4 Introduction to Divided-difference Operator and Unified Formulations

This section provides an introduction to the divided-difference operator formulation for sampled-data systems and its connection to the underlying continuous-time system, resulting in a unified formulation for sampled-data and continuous-time systems.

The relations between the state equation of a continuous-time linear time-invariant control system and its various sampled-data representations are discussed, and the notation of the divided-difference and unified formulations is introduced.

Consider the continuous-time linear time-invariant control system

$$\dot{x} = A_c x + B_c u$$

$$y = C_c x.$$

In the traditional difference-equation formulation, the corresponding sampled-data system with zero-order hold and sampling interval T is then

$$x_{k+1} = \left(e^{A_c T}\right) x_k + \left(\int_0^T e^{A_c t}\, dt\, B_c\right) u_k =: A_q x_k + B_q u_k$$

$$y_k = C_c x_k =: C_q x_k.$$

Note that $A_q \to I$ and $B_q \to 0$ as $T \to 0$. For small T, $A_q \approx I$ and information on the dynamics of the system is lost if there are round-off errors in the precision with which A_q is determined or stored. Thus, numerical difficulties arise from the use of the difference-equation formulation for implementation of fast sampling.

The corresponding divided-difference operator formulation for the sampled-data system is

$$\delta x_k := \frac{x_{k+1} - x_k}{T} = \left(\frac{A_q - I}{T}\right) x_k + \left(\frac{B_q}{T}\right) u_k =: A_\delta x_k + B_\delta u_k,$$

$$y_k = C_q x_k = C_c x_k =: C_\delta x_k.$$

Note that $\delta x_k \to \dot{x}$, $A_\delta \to A_c$, and $B_\delta \to B_c$, as $T \to 0$, eliminating the numerical difficulties associated with use of the difference-equation formulation for small T.

For a unified notation for continuous-time and sampled-data linear systems, Middleton and Goodwin [5] define

$$\rho x := \begin{cases} \dot{x} & T = 0 \text{ (continuous time)} \\ \delta x_k & T \neq 0 \text{ (sampled data)} \end{cases}$$

$$\mathcal{S}_{t_0}^{t_f} f(t)\, dt := \begin{cases} \displaystyle\int_{t_0}^{t_f} f(t)\, dt & T = 0 \\ \displaystyle T \sum_{k=\frac{t_0}{T}}^{\frac{t_f}{T}-1} f(kT) & T \neq 0 \end{cases}.$$

Note that $\mathcal{S}_{t_0}^{t_f} f(t)\, dt$ is a Riemann sum for $T \neq 0$ whenever $f(t)$ is Riemann integrable. Note that continuous functions are Riemann integrable, as are piecewise-continuous functions that have jumps only at uniformly spaced intervals. Thus, $\mathcal{S}_{t_0}^{t_f} f(t)\,dt$ is a Riemann sum if $f(t)$ is the output of a uniformly sampled linear system. In this unified formulation, the system becomes

$$\rho x = Ax + Bu$$

$$y = Cx$$

where

$$A := \begin{cases} A_c & T = 0 \\ A_\delta & T \neq 0 \end{cases}, \qquad B := \begin{cases} B_c & T = 0 \\ B_\delta & T \neq 0 \end{cases}, \qquad C = C_c \ \forall T \ \ (\text{since } C_\delta = C_c).$$

In the divided-difference operator formulation, the stability region for the poles of the system is the disc D_T of radius $1/T$ centered at $-1/T$ in the complex plane. This is derived as follows. The transform variable for the divided-difference operator form is $\gamma = (z-1)/T$, where z refers here to the standard shift-operator transform variable. (This is the notation used in [5]. In this thesis, γ is used for the unified transform variable and α is used for the H_∞ norm bound.) Then, $|z| < 1$ if and only if γ satisfies

$$\frac{T}{2}|\gamma|^2 + \Re(\gamma) < 0$$

where $\Re(\gamma)$ is the real part of γ, or, equivalently, $\gamma \in D_T$.

If $T = 0$, this reduces to $\Re(\gamma) < 0$, and D_0 is the left half-plane, the stability region for the continuous-time transform variable s. In fact, $\gamma \to s$. Thus, this definition of the stability region is also compatible with continuous-time stability.

Next, the H_∞ norm of a system is defined. Consider the specific form of the system equations

$$\rho x = Fx + Gw, \qquad x(0) = 0,$$

$$z = Hx,$$

where w is an unknown disturbance entering the system and z is the regulated-output variable, which the designer wishes to keep small. The H_∞ norm of the system is then defined as

$$\sup_{\gamma \in \partial D_T} \bar{\sigma}\left(H(\gamma I - F)^{-1}G\right),$$

where $\bar{\sigma}(\cdot)$ denotes the largest singular value of a matrix and ∂D_T denotes the boundary of the stability region D_T. Equivalently, the H_∞ norm of the system is the induced $\mathcal{L}_2/\ell_2$ norm of the system, depending on whether the continuous- or discrete-time system is considered:

$$\sup_{w \in \mathcal{L}_2/\ell_2} \frac{\|z\|_2}{\|w\|_2},$$

where $v \in \mathcal{L}_2/\ell_2$ if

$$\|v\|_2 := \left(\mathcal{S}_{t=0}^{\infty} v(t)'v(t)\, dt\right)^{\frac{1}{2}} < \infty.$$

For continuous-time systems, the set of square-integrable signals $\mathcal{L}_2$ is considered. For discrete-time systems, the set of square-summable signals ℓ_2 is considered since

$$\sup_{w \in \ell_2} \frac{\sum_{k=0}^{\infty} z(kT)'z(kT)}{\sum_{k=0}^{\infty} w(kT)'w(kT)} = \sup_{w \in \ell_2} \frac{T\sum_{k=0}^{\infty} z(kT)'z(kT)}{T\sum_{k=0}^{\infty} w(kT)'w(kT)}.$$

Discussion on the continuous-time H_∞ norm for sampled-data systems is presented in Chapter 3.

1.5 Organization

This introduction was intended to provide the requisite overview, motivation, and background for the following chapters. In Chapter 2, the unified continuous- and discrete-time results are presented. In Chapter 3, the sampled-data results are derived that bound the continuous-time norm. In Chapter 4, controller designs are given for multirate control systems. Finally, Chapter 5 concludes the thesis with a summary of the ideas developed in the thesis. Each chapter contains a discussion of numerical issues relating to the solution of the design equations and examples of the application of the controller designs developed. The proofs of the lemmas and derivations of most of the controller designs in Chapters 2–4 are in Appendices A–C, respectively.

CHAPTER 2

UNIFIED FORMULATION FOR DISCRETE- AND CONTINUOUS-TIME CONTROLLER DESIGN

In this chapter, the continuous- and discrete-time centralized and decentralized H_∞-norm-bounding controller designs and reliable controller designs are unified using the divided-difference operator/unified formulation. In addition, a unified notion of the degree of stability of a linear system is introduced, and a decentralized H_∞-norm-bounding controller design that guarantees a prescribed degree of stability for the closed-loop system is presented in the unified formulation.

The divided-difference operator formulation of the design equation for the decentralized controller permits the development of a numerical algorithm for its solution in Section 2.6.

2.1 Design Goals

This chapter contains results for two systems—a centralized system in which the entire output can be used to compute the controller and a decentralized system for which only the local part of the system output is available at each subsystem controller. In either case, the control engineer is assumed to have access to a complete description of the system for use in the controller design.

Consider first the unified formulation for the centralized system

$$\rho x = Ax + Bu + Gw_0, \qquad x(0) = 0, \tag{2.1}$$

$$y = Cx + w \tag{2.2}$$

where w_0 and w are unknown disturbances, u is the control variable, x is the state of the system, and y is the observed output. The disturbances are assumed to be continuous if $T = 0$ and piecewise constant on the sampling intervals if $T \neq 0$. (The problem in which the disturbances are continuous but $T \neq 0$ is not treated in this chapter.) Further, assume that

$$z = \begin{pmatrix} Hx \\ u \end{pmatrix} \tag{2.3}$$

is the regulated output variable, which the designer wishes to keep small.

Consider controllers of two types for the centralized system—a state-feedback controller

$$u = K^c x \tag{2.4}$$

and an observer-based feedback controller:

$$u = K^c\xi, \quad \hat{w}_0 = K^d\xi, \tag{2.5}$$

$$\rho\xi = A\xi + Bu + G\hat{w}_0 + K^o(y - C\xi). \tag{2.6}$$

The feedback form of the disturbance term in the observer is justified by comparison with the soft-constraint game problem derivation. The soft-constraint problem solution has a feedback form for the disturbance. The actual problem discussed here is a disturbance-attenuation problem, a hard-constraint problem, rather than the soft-constraint problem. The disturbance-attenuation problem solution has a mixed policy for the disturbance: the least-favorable disturbance is a linear function of a variable recursively defined, the recursion having a random variable as its initial condition [23]. Such a form would be very hard to implement in the observer, thus the disturbance is approximated in the observer by a linear feedback of the observer state, as is the worst-case in the soft-constraint problem.

Next, consider the unified formulation for a decentralized system

$$\rho x = Ax + \sum_{i=1}^{p} B_i u_i + Gw_0, \qquad x(0) = 0, \tag{2.7}$$

$$y_i = C_i x + w_i, \qquad i \in \{1, 2, ..., p\}, \tag{2.8}$$

where

$$z = \begin{pmatrix} Hx \\ u_1 \\ \vdots \\ u_p \end{pmatrix} \tag{2.9}$$

is the regulated output variable. The controller for u_i can not depend on y_j, for $j \neq i$. Consider dynamic controllers of the form

$$\rho\xi_i = A\xi_i + \sum_{j=1}^{p} B_j \hat{u}_j^i + G\hat{w}_0^i + K_i^o(y_i - C_i\xi_i) \tag{2.10}$$

$$u_i = K_i^c\xi_i, \qquad \hat{u}_j^i = K_j^c\xi_i, \qquad \hat{w}_0^i = K^d\xi_i, \tag{2.11}$$

for $i \in \{1, 2, ..., p\}$.

Denote the transfer function from the disturbance w_e to z by $T_{w_e z}(\gamma)$, where w_e denotes

$$w_0, \qquad \begin{pmatrix} w_0 \\ w \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_p \end{pmatrix},$$

depending on whether state feedback, observer feedback, or the decentralized case is considered.

The H_∞ norm of the closed-loop system is then

$$\|T_{w_e z}\|_\infty := \sup_{\gamma=\frac{e^{j\theta}-1}{T}} \bar{\sigma}(T_{w_e z}(\gamma)),$$

where $\bar{\sigma}(\cdot)$ denotes the largest singular value of a matrix, or, equivalently,

$$\sup_{w_e \in \mathcal{L}_2} \frac{\|z\|_2}{\|w_e\|_2},$$

where $v \in \mathcal{L}_2$ if

$$\|v(t)\|_2 := \left(\mathcal{S}_{t=0}^{\infty} v(t)'v(t)\,dt\right)^{\frac{1}{2}} < \infty,$$

as discussed in Chapter 1.

The goal in this chapter is to unify the continuous- and discrete-time approaches to the design of controllers of the above forms that solve the H_∞-norm-bounding and reliable control problems for sampled-data linear systems and the underlying continuous-time systems.

2.2 Bounded Real Lemma

The crucial lemma in the development of all of the designs in this chapter is the following bounded real lemma, which is a variation of Willems' result from [19].

Lemma 2.2.1 *Consider a linear system T_{wz} with a detectable realization*

$$\rho x = Fx + Gw, \qquad x(0) = 0,$$

$$z = Hx.$$

If there exist a real, symmetric matrix $X \geq 0$ and a real $\alpha > 0$ such that

(i) $F'XL^{-1} + XL^{-1}F + TF'XL^{-1}F + \frac{1}{\alpha^2}XGG'XL^{-1} + H'H \leq 0$

(ii) $\alpha^2 I - TG'XG > 0$

where $L := I - T\frac{1}{\alpha^2}GG'X$, then

(a) the eigenvalues of F lie in D_T, the stability region for sampling interval T

(b) $\|T_{wz}\|_\infty \leq \alpha$.

Note that the term $\frac{1}{\alpha^2}XGG'XL^{-1}$ is symmetric since it can be rearranged as

$$XGG'X(I + TGG'X)^{-1} = X(I + TGG'X)^{-1}GG'X = (I + TXGG')^{-1}XGG'X,$$

which is its transpose if X is symmetric.

2.3 Centralized Discrete- and Continuous-time Controller Design

In this section, the centralized designs are presented for the state-feedback and observer-feedback controllers for both the H_∞-norm-bounding and reliable problems. The derivations are given in Appendix A.2.

2.3.1 State feedback

The most direct result to follow from the bounded real lemma is the state-feedback result for the centralized system.

Theorem 2.3.1 *For the centralized system (2.1), (2.3), with (A, H) detectable and with state-feedback (2.4), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_0z}\|_\infty \leq \alpha$ is*

$$K^c = -B'X\Lambda^{-1}(I + TA)$$

where

$$\Lambda := I + T(BB' - \frac{1}{\alpha^2}GG')X$$

and $X \geq 0$ satisfies

(i) $A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(BB' - \frac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H = 0$

(ii) $\alpha^2 I - TG'XG > 0.$

The term $X(BB' - \frac{1}{\alpha^2}GG')X\Lambda^{-1}$ is also symmetric.

Proof The proof is provided in Appendix A.2. □

2.3.2 Observer-based feedback

Now consider the full centralized system with output y (2.1), (2.2), (2.3), together with the proposed form of output observer (2.5), (2.6). The following theorem gives sufficient conditions for stability and disturbance attenuation for this case.

Theorem 2.3.2 *For the centralized system (2.1), (2.2), (2.3), with (A, H) detectable and with observer-based feedback (2.5), (2.6), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_ez}\|_\infty \leq \alpha$ is*

$$K^c = -B'X\Lambda^{-1}(I + TA), \qquad K^d = \frac{1}{\alpha^2}G'X\Lambda^{-1}(I + TA),$$

$$K^o = (I - \frac{1}{\alpha^2}YX)^{-1}(I + TA)\Pi^{-1}YC', \tag{2.12}$$

where

$$\Lambda := I + T(BB' - \frac{1}{\alpha^2}GG')X, \qquad \Pi := I + TY(C'C - \frac{1}{\alpha^2}H'H), \tag{2.13}$$

and $X \geq 0$ *and* $Y > 0$ *satisfy*

$$A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(BB' - \frac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H = 0 \tag{2.14}$$

$$A\Pi^{-1}Y + \Pi^{-1}YA' + TA\Pi^{-1}YA' - \Pi^{-1}Y(C'C - \frac{1}{\alpha^2}H'H)Y + GG' = 0 \tag{2.15}$$

such that

$$\rho(YX) < \alpha^2 \tag{2.16}$$

and the eigenvalues of $A + GK^d - K^oC$ *lie in* D_T*, and*

$$\Xi(X,Y) := \begin{pmatrix} \alpha^2 I - TG'\alpha^2 Y^{-1}G & TG'(\alpha^2 Y^{-1} - X)K^o \\ TK^{o\prime}(\alpha^2 Y^{-1} - X)G & \alpha^2 I - TK^{o\prime}(\alpha^2 Y^{-1} - X)K^o \end{pmatrix} > 0 \tag{2.17}$$

(or

$$X > 0, \qquad \alpha^2 I - TG'XG > 0,$$

$$|L^{-1}(I + TA) - TK^oC| \neq 0,$$

and

$$\alpha^2 Y^{-1} > (I + TA)'XL^{-1}(I + TA) + TH'H\,).$$

2.3.3 Reliability to sensor and actuator outages

2.3.3.1 Reliability to sensor outages

Now we alter the basic observer-based design to guarantee reliability in the presence of sensor outages. This design will perform with stability and a given H_∞-norm bound even when any number of sensors from a prespecified set fail. Clearly, there may be some sensors that are essential to the operation of the controller. No design can be reliable to failures of those sensors. Designers should consider adding redundancy of sensors for those that are critical for operation. There may also be conservatism in any design that aims at performance in the presence of sensor failures.

Now assume that stability and a given H_∞-norm bound are required despite subsets of failures in a given set of susceptible sensors. Suppose that the sensors corresponding to certain y_j are susceptible. Let $\Omega \subset \{1, 2, ..., \dim(y)\}$ be the index set of the susceptible y_j's. We require performance, i.e., stability and disturbance attenuation, during the failure of any subset of susceptible sensors, $\omega \subseteq \Omega$. We assume here that failed sensors have zero output with no noise or bias. Given the system (2.1), (2.2), (2.3), with the observer (2.5), (2.6), define the following matrices:

- C_Ω as C with the rows not in Ω set equal to zero
- C_ω as C with the rows not in ω set equal to zero
- K^o_ω as K^o with the columns not in ω set equal to zero
- $C_{\bar{\Omega}} := C - C_\Omega$ • $C_{\bar{\omega}} := C - C_\omega$ • $K^o_{\bar{\omega}} := K^o - K^o_\omega$.

Thus, $C_\omega{}'C_\omega \leq C_\Omega{}'C_\Omega$, and $K^oC = K^o_\omega C_\omega + K^o_{\bar{\omega}} C_{\bar{\omega}}$.

The following theorem gives the design equations for the reliable centralized case, for possible sensor outages.

Theorem 2.3.3 *For the centralized system (2.1), (2.2), (2.3), with (A, H) detectable and with output feedback (2.5), (2.6),a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_e z}\|_\infty \leq \alpha$ for all sensor failures $\omega \subseteq \Omega$ is*

$$K^c = -B'X\Lambda^{-1}(I + TA), \qquad K^d = \frac{1}{\alpha^2}G'X\Lambda^{-1}(I + TA),$$

$$K^o = (I - \frac{1}{\alpha^2}YX)^{-1}(I + TA)\Pi^{-1}YC'$$

where

$$\Lambda := I + T(BB' - \frac{1}{\alpha^2}GG')X, \qquad \Pi := I + TY(C_{\bar{\Omega}}{}'C_{\bar{\Omega}} - \frac{1}{\alpha^2}H'H)$$

and $X \geq 0$ and $Y > 0$ satisfy

$$A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(BB' - \frac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H + \alpha^2 C_\Omega{}'C_\Omega = 0 \qquad (2.18)$$

$$A\Pi^{-1}Y + \Pi^{-1}YA' + TA\Pi^{-1}YA' - \Pi^{-1}Y(C_{\bar{\Omega}}{}'C_{\bar{\Omega}} - \frac{1}{\alpha^2}H'H)Y + GG' = 0 \qquad (2.19)$$

such that

$$\rho(YX) < \alpha^2$$

and the eigenvalues of $A + GK^d - K^oC$ lie in D_T, and

$$\Xi(X, Y) > 0$$

and, for all $\omega \subseteq \Omega$,

$$\bar{\Xi}(X,Y) := \begin{pmatrix} \alpha^2 I - TG'\alpha^2 Y^{-1} G & TG'(\alpha^2 Y^{-1} - X)K^o_{\bar{\omega}} \\ TK^{o\prime}_{\bar{\omega}}(\alpha^2 Y^{-1} - X)G & \alpha^2 I - TK^{o\prime}_{\bar{\omega}}(\alpha^2 Y^{-1} - X)K^o_{\bar{\omega}} \end{pmatrix} > 0 \tag{2.20}$$

(or, instead of the conditions on $\Xi(X,Y)$ and $\bar{\Xi}(X,Y)$,

$$X > 0, \qquad \alpha^2 I - TG'XG > 0,$$

$$|L^{-1}(I + TA) - TK^o C| \neq 0,$$

and

$$\alpha^2 Y^{-1} > (I + TA)'XL^{-1}(I + TA) + TH'H + T\alpha^2 C_\Omega{}' C_\Omega\,).$$

Note that the definition of Π and the two Riccati equations are different from the standard case.

2.3.3.2 Reliability to actuator outages

Now suppose that a controller design is required that achieves stability and a given H_∞-norm bound even when actuators in a given susceptible subset fail. Assume that failed actuators have zero output. Let

$$\Omega \subset \{1, 2, ..., dim(u)\}$$

be the index set of susceptible actuator inputs u_j, and define

- B_Ω as B with the columns not in Ω set equal to zero
- B_ω as B with the columns not in ω set equal to zero
- K^c_Ω as K with the rows not in Ω set equal to zero
- K^c_ω as K with the rows not in ω set equal to zero
- $B_{\bar{\Omega}} := B - B_\Omega$ • $B_{\bar{\omega}} := B - B_\omega$
- $K^c_{\bar{\Omega}} := K^c - K^c_\Omega$ • $K^c_{\bar{\omega}} := K^c - K^c_\omega$.

Thus, $B_\omega B_\omega{}' \leq B_\Omega B_\Omega{}'$, and $BK^c = B_\Omega K^c_\Omega + B_{\bar{\Omega}} K^c_{\bar{\Omega}}$.

The following theorem gives the design equations for the reliable centralized case, for possible actuator outages. A modification to the observer form has been made to account for possible actuator outages.

Theorem 2.3.4 *For the centralized system (2.1), (2.2), (2.3), with (A, H) detectable and with output feedback*

$$u = K^c\xi, \quad \hat{w}_0 = K^d\xi, \quad \hat{u}_\Omega = K^\Omega\xi, \tag{2.21}$$

$$\rho\xi = A\xi + Bu + G\hat{w}_0 - B_\Omega\hat{u}_\Omega + K^o(y - C\xi), \tag{2.22}$$

a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_e z}\|_\infty \leq \alpha$ for all actuator failures $\omega \subseteq \Omega$ is

$$K^c = -B'X\Lambda^{-1}(I + TA), \qquad K^d = \frac{1}{\alpha^2}G'X\Lambda^{-1}(I + TA),$$

$$K^\Omega = -B_\Omega{}'X\Lambda^{-1}(I + TA),$$

$$K^o = (I - \frac{1}{\alpha^2}YX)^{-1}(I + TA)\Pi^{-1}YC'$$

where

$$\Lambda := I + T(BWbB_{\bar{\Omega}}{}' - \frac{1}{\alpha^2}GG')X, \qquad \Pi := I + TY(C'C - \frac{1}{\alpha^2}H'H)$$

and $X \geq 0$ and $Y > 0$ satisfy

$$A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(B_{\bar{\Omega}}B_{\bar{\Omega}}{}' - \frac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H = 0 \tag{2.23}$$

$$A\Pi^{-1}Y + \Pi^{-1}YA' + TA\Pi^{-1}YA' - \Pi^{-1}Y(C'C - \frac{1}{\alpha^2}H'H)Y + GG' + \alpha^2B_\Omega B_\Omega{}' = 0 \tag{2.24}$$

such that

$$\rho(YX) < \alpha^2$$

and the eigenvalues of $A + GK^d - K^oC + BK^c$ lie in D_T and

$$I - T(B_\Omega{}' \;\; - B_\Omega{}')X_eL_e{}^{-1}\begin{pmatrix} B_\Omega \\ -B_\Omega{}' \end{pmatrix} > 0$$

(or

$$X > 0, \qquad X^{-1} - T\frac{1}{\alpha^2}GG' - TB_\Omega B_\Omega{}' > 0,$$

$$|(L - TB_\Omega B_\Omega{}'X)^{-1}(I + TA) - TK^oC| \neq 0,$$

and

$$\alpha^2Y^{-1} > (I + TA)'X(L - TB_\Omega B_\Omega{}'X)^{-1}(I + TA) + TH'H\,).$$

2.4 Decentralized Discrete- and Continuous-time Controller Design

Now consider the decentralized system (2.7), (2.8), with the dynamic controller structure (2.10), (2.11). We first give a control design that will guarantee stability and an H_∞-norm bound for the closed-loop system. Then, we modify the basic decentralized results to guarantee performance despite outages in the subsystem controllers.

2.4.1 Basic decentralized results

The decentralized system can be written in the shorthand notation

$$\rho x = Ax + Bu + Gw_0, \qquad x(0) = 0,$$

$$y = Cx + w, \qquad z = \begin{pmatrix} Hx \\ u \end{pmatrix}$$

where

$$u := \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{pmatrix}, \qquad y := \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}, \qquad w := \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_p \end{pmatrix}$$

$$B := (B_1 \; B_2 \cdots B_p), \qquad C := \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_p \end{pmatrix}.$$

Further, define the following composite matrices:

$$B_D := \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & B_p \end{pmatrix}, \qquad C_D := \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & C_p \end{pmatrix},$$

$$K_D^o := \begin{pmatrix} K^o{}_1 & 0 & \cdots & 0 \\ 0 & K^o{}_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K^o{}_p \end{pmatrix}, \qquad G_C := \begin{pmatrix} G \\ G \\ \vdots \\ G \end{pmatrix}, \qquad B_C := \begin{pmatrix} B \\ B \\ \vdots \\ B \end{pmatrix},$$

$$\mathcal{X}_D := \begin{pmatrix} \mathcal{X} & 0 & \cdots & 0 \\ 0 & \mathcal{X} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathcal{X} \end{pmatrix}, \qquad \text{where } \mathcal{X} := X\Lambda^{-1}(I+TA),$$

and

$$A_e \;:=\; \begin{pmatrix} A-(BB'-\frac{1}{\alpha^2}GG')\mathcal{X} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A-(BB'-\frac{1}{\alpha^2}GG')\mathcal{X} \end{pmatrix}$$

$$+\begin{pmatrix} B_1B_1'\mathcal{X} & B_2B_2'\mathcal{X} & \cdots & B_pB_p'\mathcal{X} \\ B_1B_1'\mathcal{X} & \ddots & & \vdots \\ \vdots & & \ddots & B_pB_p'\mathcal{X} \\ B_1B_1'\mathcal{X} & \cdots & B_{p-1}B_{p-1}'\mathcal{X} & B_pB_p'\mathcal{X} \end{pmatrix}.$$

Then the following theorem gives the design conditions for the decentralized system.

Theorem 2.4.1 *For the decentralized system (2.7), (2.8), with (A,H) detectable and with observer-based feedback (2.10), (2.11), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_e z}\|_\infty \le \alpha$ is*

$$K_i^c = -B_i'\mathcal{X}, \quad i \in \{1,2,...,p\}, \qquad K^d = \frac{1}{\alpha^2}G'\mathcal{X}, \tag{2.25}$$

where

$$\mathcal{X} := X\Lambda^{-1}(I+TA), \qquad \Lambda := I + T(BB'-\frac{1}{\alpha^2}GG')X, \qquad L := I - T\frac{1}{\alpha^2}GG'X \tag{2.26}$$

and K_D^o block diagonal, $X \ge 0$, and $W > 0$ satisfy

$$A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(BB'-\frac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H = 0 \tag{2.27}$$

$$\begin{aligned} &A_fW + WA_f' + TA_fWA_f' + \tfrac{1}{\alpha^2}G_C(I - T\tfrac{1}{\alpha^2}G'XG)^{-1}G_C' + \tfrac{1}{\alpha^2}K_D^oK_D^{o\prime} \\ &\quad + (I+TA_f)W\mathcal{X}_D'B_D\left((I+TB'XL^{-1}B)^{-1} - TB_D'\mathcal{X}_DW\mathcal{X}_D'B_D\right)^{-1} \\ &\quad\; \cdot B_D'\mathcal{X}_DW(I+TA_f)' \\ &\quad + TK_D^oC_D(W^{-1} - T\mathcal{X}_D'B_D(I+TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}C_D'K_D^{o\prime} \\ &\quad - K_D^oC_D(W^{-1} - T\mathcal{X}_D'B_D(I+TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}(I+TA_f)' \\ &\quad - (I+TA_f)(W^{-1} - T\mathcal{X}_D'B_D(I+TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}C_D'K_D^{o\prime} = 0 \end{aligned} \tag{2.28}$$

where

$$A_f(X) := A_e + T\frac{1}{\alpha^2}G_C G' X L^{-1} B B_D' \mathcal{X}_D, \tag{2.29}$$

such that the eigenvalues of $A_e - K_D^o C_D$ *are in the stability region and either*

$$\Xi_D(X,W) := \begin{pmatrix} \alpha^2 I - TG'XG - TG_C' W^{-1} G_C & TG_C' W^{-1} K_D^o \\ TK_D^{o\prime} W^{-1} G_C & \alpha^2 I - TK_D^{o\prime} W^{-1} K_D^o \end{pmatrix} > 0 \tag{2.30}$$

or

$$X > 0, \qquad \alpha^2 I - TG'XG > 0, \qquad |I + TA_f - TK_D^o C_D| \neq 0,$$

$$W^{-1} > T\mathcal{X}_D' B_D (I + TB'XL^{-1}B) B_D' \mathcal{X}_D.$$

The choice of $K_D^o = \alpha^2 W_D C_D'$, where W_D is the block diagonal part of W, results in one solution.

It is not known yet whether conditions can be found that guarantee the existence of a solution to the design equations for large enough α.

2.4.2 Reliable decentralized results

Now we will modify the basic decentralized design to produce a design that is reliable to subsystem outages. The subsystems can fail in a variety of ways. We will consider failures of two types: $y_i = 0$ or $u_i = 0$ for each susceptible subsystem i. Let $\Omega \subset \{1, 2, ..., p\}$ be the set of indices of the susceptible subsystems, and let $\omega \subseteq \Omega$ be the set of indices of the subsystems that actually experience failures.

Define the following matrices:

- C_Ω as C with the blocks not in Ω set equal to zero
- C_ω as C with the blocks not in ω set equal to zero
- $K_{D,\Omega}^o$ as K_D^o with diagonal blocks not in Ω set equal to zero
- $K_{D,\omega}^o$ as K_D^o with diagonal blocks not in ω set equal to zero
- B_Ω as B with the blocks not in Ω set equal to zero
- B_ω as B with the blocks not in ω set equal to zero
- $B_{D,\Omega}$ as B_D with the diagonal blocks not in Ω set equal to zero
- $B_{D,\omega}$ as B_D with the diagonal blocks not in ω set equal to zero

- $B_{C,\Omega}$ as B_C with the columns of blocks not in Ω set equal to zero
- $B_{C,\omega}$ as B_C with the columns of blocks not in ω set equal to zero
- $K^o_{D,\bar{\omega}} := K^o_D - K^o_{D,\omega}$ • $B_{D,\bar{\omega}} := B_D - B_{D,\omega}$
- $B_{\bar{\Omega}} := B - B_\Omega$ • $B_{\bar{\omega}} := B - B_\omega$.

The following theorem gives the controller design for the case in which subsystem sensor failures may occur (i.e., when $y_i = 0 \ \forall i \in \omega \subseteq \Omega$).

Theorem 2.4.2 *For the decentralized system (2.7), (2.8), with (A, H) detectable and with observer-based feedback (2.10), (2.11), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_e z}\|_\infty \leq \alpha$ for all subsystem sensor failures $\omega \subseteq \Omega$ is*

$$K_i^c = -B_i'\mathcal{X}, \quad i \in \{1, 2, ..., p\}, \qquad K^d = \frac{1}{\alpha^2}G'\mathcal{X}, \tag{2.31}$$

where

$$\Lambda := I + T(BB' - \frac{1}{\alpha^2}GG')X, \qquad L := I - T\frac{1}{\alpha^2}GG'X \tag{2.32}$$

and K_D^o block diagonal, $X \geq 0$, and $W > 0$ satisfy

$$\begin{aligned} &A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(BB' - \tfrac{1}{\alpha^2}GG')X\Lambda^{-1} \\ &+ H'H + \alpha^2 C_\Omega' C_\Omega = 0 \end{aligned} \tag{2.33}$$

$$\begin{aligned} A_fW &+ WA_f' + TA_fWA_f' + \tfrac{1}{\alpha^2}G_C(I - T\tfrac{1}{\alpha^2}G'XG)^{-1}G_C' + \tfrac{1}{\alpha^2}K_D^oK_D^{o\prime} \\ &+ (I + TA_f)W\mathcal{X}_D'B_D\left((I + TB'XL^{-1}B)^{-1} - TB_D'\mathcal{X}_DW\mathcal{X}_D'B_D\right)^{-1} \\ &\quad \cdot B_D'\mathcal{X}_DW(I + TA_f)' \\ &+ TK_D^oC_D(W^{-1} - T\mathcal{X}_D'B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}C_D'K_D^{o\prime} \\ &- K_D^oC_D(W^{-1} - T\mathcal{X}_D'B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}(I + TA_f)' \\ &- (I + TA_f)(W^{-1} - T\mathcal{X}_D'B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}C_D'K_D^{o\prime} = 0 \end{aligned} \tag{2.34}$$

where

$$A_f(X) := A_e + T\frac{1}{\alpha^2}G_CG'XL^{-1}BB_D'\mathcal{X}_D, \tag{2.35}$$

such that the eigenvalues of $A_e - K_D^oC_D$ are in the stability region and either both $\Xi_D(X, W) > 0$ and, $\forall \omega \subseteq \Omega$,

$$\bar{\Xi}_D(X, W) := \begin{pmatrix} \alpha^2 I - TG'XG - TG_C'W^{-1}G_C & TG_C'W^{-1}K^o_{D,\bar{\omega}} \\ TK^{o\prime}_{D,\bar{\omega}}W^{-1}G_C & \alpha^2 I - TK^{o\prime}_{D,\bar{\omega}}W^{-1}K^o_{D,\bar{\omega}} \end{pmatrix} > 0 \tag{2.36}$$

or

$$X > 0, \qquad \alpha^2 I - TG'XG > 0, \qquad |I + TA_f - TK^o_D C_D| \neq 0,$$
$$W^{-1} > T\mathcal{X}'_D B_D (I + TB'XL^{-1}B) B'_D \mathcal{X}_D.$$

As before, the choice of $K^o_D = \alpha^2 W_D C'_D$, where W_D is the block diagonal part of W, results in one solution.

The next theorem provides the controller design for the decentralized case with subsystem actuator outages of the form $u_i = 0\ \forall i \in \omega \subseteq \Omega$.

Theorem 2.4.3 *For the decentralized system (2.7), (2.8), with (A, H) detectable and with observer-based feedback*

$$\rho\xi_i = A\xi_i + \sum_{j \in \bar{\Omega}} B_j \hat{u}^i_j + G\hat{w}^i_0 + K^o{}_i(y_i - C_i\xi_i) \tag{2.37}$$
$$u_i = K^c_i \xi_i, \qquad \hat{u}^i_j = K^c_j \xi_i, \qquad \hat{w}^i_0 = K^d \xi_i,$$

for $i \in \{1, 2, ..., p\}$, a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_e z}\|_\infty \leq \alpha$ for all $\omega \subseteq \Omega$ is

$$K^c_i = -B'_i \mathcal{X}, \quad i \in \{1, 2, ..., p\}, \qquad K^d = \frac{1}{\alpha^2} G'\mathcal{X},$$

where

$$\Lambda := I + T(B_{\bar{\Omega}} B'_{\bar{\Omega}} - \frac{1}{\alpha^2} GG')X, \qquad L := I - T\frac{1}{\alpha^2} GG'X \tag{2.38}$$

and K^o_D block diagonal, $X > 0$, and $W > 0$ satisfy

$$A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(B_{\bar{\Omega}} B'_{\bar{\Omega}} - \frac{1}{\alpha^2} GG')X\Lambda^{-1} + H'H = 0 \tag{2.39}$$

$$\begin{aligned}
&A_f W + WA'_f + TA_f WA'_f + \tfrac{1}{\alpha^2} K^o_D K^{o\prime}_D \\
&\quad + \tfrac{1}{\alpha^2}(G_C \ \alpha B_{C,\Omega})\left(I - T\tfrac{1}{\alpha^2}\left(\begin{smallmatrix} G' \\ \alpha B'_\Omega \end{smallmatrix}\right) X (G \ \alpha B_\Omega)\right)^{-1}\left(\begin{smallmatrix} G'_C \\ \alpha B'_{C,\Omega} \end{smallmatrix}\right) \\
&\quad + (I + TA_f) W \mathcal{X}'_D B_D \left((I + TB'XL^{-1}B)^{-1} - TB'_D \mathcal{X}_D W \mathcal{X}'_D B_D\right)^{-1} \\
&\quad \cdot B'_D \mathcal{X}_D W (I + TA_f)' \\
&\quad + TK^o_D C_D (W^{-1} - T\mathcal{X}'_D B_D (I + TB'XL^{-1}B) B'_D \mathcal{X}_D)^{-1} C'_D K^{o\prime}_D \\
&\quad - K^o_D C_D (W^{-1} - T\mathcal{X}'_D B_D (I + TB'XL^{-1}B) B'_D \mathcal{X}_D)^{-1} (I + TA_f)' \\
&\quad - (I + TA_f)(W^{-1} - T\mathcal{X}'_D B_D (I + TB'XL^{-1}B) B'_D \mathcal{X}_D)^{-1} C'_D K^{o\prime}_D = 0
\end{aligned} \tag{2.40}$$

where

$$\begin{aligned}
A_f(X) \quad := \quad & A_e + T\tfrac{1}{\alpha^2}(G_C \ \alpha B_{C,\Omega})\left(\begin{smallmatrix} G' \\ \alpha B'_\Omega \end{smallmatrix}\right) X (L - TB_\Omega B'_\Omega X)^{-1} BB'_D \mathcal{X}_D \\
& + B_{D,\Omega} B'_{D,\Omega} \mathcal{X}_D + TB_{C,\Omega} B'_\Omega X (L - TB_\Omega B'_\Omega X)^{-1} BB'_D \mathcal{X}_D
\end{aligned} \tag{2.41}$$

such that the eigenvalues of $A_e - B_C B'_D \mathcal{X}_D - K^o_D C_D$ *are in* D_T *and*

$$I - T(B'_\Omega \;\; - B'_{C,\Omega})X_e L_e{}^{-1} \begin{pmatrix} B_\Omega \\ -B_{C,\Omega} \end{pmatrix} > 0$$

or

$$X^{-1} - T\frac{1}{\alpha^2}GG' - TB_\Omega B'_\Omega > 0, \qquad |I + TA_f - TK^o_D C_D| \neq 0,$$

$$W^{-1} > T\mathcal{X}'_D B_D[I + TB'X(L - TB_\Omega B'_\Omega X)^{-1}B]B'_D\mathcal{X}_D.$$

Again, the choice of $K^o_D = \alpha^2 W_D C'_D$, where W_D is the block diagonal part of W, results in one solution.

2.5 H_∞-norm-bounding Control with a Prescribed Degree of Stability

In this section, a unified definition for the degree of stability of continuous- and sampled-data systems is given, and an H_∞-norm-bounding decentralized controller design is developed, in the unified formulation, that guarantees a prescribed degree of stability.

2.5.1 Unified notion of degree of stability

A continuous-time linear system is said to have degree of stability η if all of the poles of the system satisfy $\Re(s) \leq -\eta$ [24].

Discrete-time systems could be defined to have degree of stability μ, where $\mu < 1$, if all of the poles of the discrete-time system are inside the disk of radius $1 - \mu$ centered at 0 in the z-plane. However, for sampled-data systems with sampling period $1/T$, that corresponds to having poles in the γ-transform plane in a disc of radius $(1-\mu)/T$ centered at $-1/T$. This is equivalent to the condition

$$\frac{T}{2}|\gamma|^2 + \Re(\gamma) < \frac{\mu^2 - 2\mu}{2T} < 0.$$

If $T \to 0$, this approaches $\Re(\gamma) < -\infty < 0$. Thus, as the sampling rate increases, the required degree of stability on the underlying system increases.

To avoid this inconsistency, we define the degree of stability for a sampled-data system based on the degree of stability of the underlying continuous-time system as follows.

Definition 2.5.1 *The linear system* $\rho x = Fx$ *is said to have* **degree of stability** η, *where* $0 < \eta < 1/T$, *if*

$$\frac{T}{2}|\gamma|^2 + \Re(\gamma) < -\eta + \frac{T}{2}\eta^2.$$

Note that, if $T = 0$, this reduces to the condition $\Re(s) \leq -\eta$, which is the definition for continuous-time systems. If $T \neq 0$, this definition is equivalent to requiring that the poles lie in D_T^η, a disc of radius $(1/T) - \eta > 0$ centered at $-1/T$ in the γ-transform plane.

2.5.2 Modified bounded real lemma—prescribed degree of stability

First, Lemma 2.2.1 is modified to guarantee an H_∞-norm bound and a prescribed degree of stability if the conditions of the lemma are satisfied, as follows.

Lemma 2.5.1 *Consider a linear system T_{wz} with a realization*

$$\rho x = Fx + Gw, \qquad x(0) = 0,$$

$$z = Hx$$

with all unobservable modes of F in D_T^η. If there exist a real, symmetric matrix $X \geq 0$ and a real $\alpha > 0$ such that

(i) $F'XL^{-1} + XL^{-1}F + TF'XL^{-1}F + \frac{1}{\alpha^2}XGG'XL^{-1} + H'H \leq -(2 - T\eta)\eta X$

(ii) $\alpha^2 I - TG'XG > 0$

where $L := I - T\frac{1}{\alpha^2}GG'X$, *then*

(a) the eigenvalues of F lie in D_T^η

(b) $\|T_{wz}\|_\infty \leq \alpha$.

Note that no extra restrictions are required on the disturbances since the derivations of the controllers with a degree of stability are based on Lemma 2.5.1, not on a direct modification of the controller designs derived before, and the proof of Lemma 2.5.1 does not require any restrictions on the disturbances within the set of $\mathcal{L}_2/\ell_2$ signals.

2.5.3 Decentralized control with a prescribed degree of stability

Theorem 2.4.1 is modified in the following theorem to guarantee a prescribed degree of stability for the resulting closed-loop system.

Theorem 2.5.1 *For the decentralized system (2.7), (2.8), with (A, H) having all unobservable modes in D_T^η and with observer-based feedback (2.10), (2.11), a sufficient condition to guarantee that the closed-loop plant has degree of stability η and that $\|T_{w_e z}\|_\infty \leq \alpha$ is*

$$K_i^c = -B_i'\mathcal{X}, \quad i \in \{1, 2, ..., p\}, \qquad K^d = \frac{1}{\alpha^2}G'\mathcal{X},$$

where

$$\mathcal{X} := X\Lambda^{-1}(I+TA), \qquad \Lambda := I + T(BB' - \frac{1}{\alpha^2}GG')X, \qquad L := I - T\frac{1}{\alpha^2}GG'X$$

and K_D^o block diagonal, $X \geq 0$, and $W > 0$ satisfy

$$A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(BB' - \frac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H = -(2-T\eta)\eta X \qquad (2.42)$$

$$\begin{aligned} W = {} & [(I+TA_f) - TK_D^oC_D]((1-T\eta)^2W^{-1} - T\mathcal{X}_D'B_D(I+TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1} \\ & \cdot [(I+TA_f) - TK_D^oC_D]' \\ & + T\tfrac{1}{\alpha^2}G_C(I - T\tfrac{1}{\alpha^2}G'XG)^{-1}G_C' + T\tfrac{1}{\alpha^2}K_D^oK_D^{o\prime}. \end{aligned} \qquad (2.43)$$

where

$$A_f(X) := A_e + T\frac{1}{\alpha^2}G_CG'XL^{-1}BB_D'\mathcal{X}_D,$$

such that the eigenvalues of $A_e - K_D^oC_D$ are in D_T^η and either

$$\Xi_D(X,W) := \begin{pmatrix} \alpha^2I - TG'XG - TG_C'W^{-1}G_C & TG_C'W^{-1}K_D^o \\ TK_D^{o\prime}W^{-1}G_C & \alpha^2I - TK_D^{o\prime}W^{-1}K_D^o \end{pmatrix} > 0$$

or

$$X > 0, \qquad \alpha^2I - TG'XG > 0, \qquad |I + TA_f - TK_D^oC_D| \neq 0,$$

$$(1-T\eta)^2W^{-1} > T\mathcal{X}_D'B_D(I+TB'XL^{-1}B)B_D'\mathcal{X}_D.$$

Proof The differences from Theorem 2.4.1 are found by setting $X_e = \begin{pmatrix} X & 0 \\ 0 & X_1 \end{pmatrix}$, as before, and requiring condition (i) of Lemma 2.5.1 to hold with equality. □

2.6 Numerical Issues

The centralized theorem's design equations are simply the unified form of algebraic Riccati equations. Each can be solved in a single step by finding the eigenvalues and eigenvectors of the appropriate Hamiltonian matrix.

The associated Hamiltonian [5] for the state algebraic Riccati equation is

$$M_X = \begin{pmatrix} A + T(BB' - \frac{1}{\alpha^2}GG')(I+TA)'^{-1}H'H & -(BB' - \frac{1}{\alpha^2}GG')(I+TA)'^{-1} \\ -(I+TA)'^{-1}H'H & -(I+TA)'^{-1}A' \end{pmatrix}.$$

If

$$M_X \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \Sigma_s$$

where Σ_s is a diagonal matrix containing the stable eigenvalues of M_X, then the solution to the Riccati equation is $X = V_2 V_1^{-1}$.

The Hamiltonian matrix corresponding to the output algebraic Riccati equation is

$$M_Y = \begin{pmatrix} A' + T(C'C - \frac{1}{\alpha^2}H'H)(I+TA)^{-1}GG' & -(C'C - \frac{1}{\alpha^2}H'H)(I+TA)^{-1} \\ -(I+TA)^{-1}GG' & -A(I+TA)^{-1} \end{pmatrix}.$$

The reliable centralized cases can be solved by making the appropriate changes to the Hamiltonians.

The decentralized case, on the other hand, does not result in an algebraic Riccati equation form. For the continuous-time decentralized case, Veillette [25] solves for W by treating the design Equation (A.4) as a Riccati equation with an extra nonnegative definite term added to it. He iteratively solves for W, first evaluating the final term and then solving for a new value of W from the appropriate Hamiltonian, viewing the last term as fixed.

Paz [2] was not able to solve numerically the discrete-time design equations for a positive-definite (stabilizing) solution.

The decentralized equations in this thesis differ from his and are amenable to solution because the direction of the iteration is reversed. Note that the Riccati difference equations that arise from the finite-horizon centralized problem, or from the linear quadratic regulator problem, progress backward in time for the state Riccati equation and *forward* in time for the output Riccati equation. Following this sort of development for the decentralized case, we discover that the X_1 equation is backward in time and that only by performing the inversion do we arrive at the forward equation in terms of $X_1{}^{-1}$, or W.

Furthermore, since the divided-difference design equations have solutions close to the continuous-time solutions for small T, the logical initial value for the iteration is the continuous-time solution.

Numerical experience shows that, starting at the continuous-time solution for W, the difference equation

$$\begin{aligned} \rho W = {} & A_f W + W A_f' + T A_f W A_f' + \tfrac{1}{\alpha^2} G_C (I - T\tfrac{1}{\alpha^2} G'XG)^{-1} G_C' \\ & + (I + TA_f) W \mathcal{X}_D' B_D \left((I + TB'XL^{-1}B)^{-1} - TB_D' \mathcal{X}_D W \mathcal{X}_D' B_D \right)^{-1} \\ & \cdot B_D' \mathcal{X}_D W (I + TA_f)' + \tfrac{1}{\alpha^2} K_D^o K_D^{o\prime} \\ & - K_D^o C_D (W^{-1} - T\mathcal{X}_D' B_D (I + TB'XL^{-1}B) B_D' \mathcal{X}_D)^{-1} (I + TA_f)' \\ & - (I + TA_f)(W^{-1} - T\mathcal{X}_D' B_D (I + TB'XL^{-1}B) B_D' \mathcal{X}_D)^{-1} C_D' K_D^{o\prime} \\ & + T K_D^o C_D (W^{-1} - T\mathcal{X}_D' B_D (I + TB'XL^{-1}B) B_D' \mathcal{X}_D)^{-1} C_D' K_D^{o\prime} \end{aligned}$$

converges to a positive-definite stabilizing solution. There is a limited region of initial solutions for which the difference equation converges, but for small T the continuous-time solution lies in the region of convergence. Increasing T by small steps, the solution can be obtained for larger T.

The resulting error from the numerical algorithm in the solution is small but indefinite, and thus the expression (i) in Lemma 2.2.1 is indefinite, rather than negative semi-definite. A modification was made for computational purposes to the algorithms to guarantee that the resulting values satisfy the inequality (i) strictly (setting the left-hand side equal to $-\varepsilon I$):

$$A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(BB' - \frac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H = -\varepsilon I,$$

$$\begin{aligned}
\rho W = {} & A_f W + WA_f' + TA_f WA_f' + \varepsilon(I + TA_f)W(I - \varepsilon TW)^{-1}W(I + TA_f)' \\
& + (I + TA_f)(W + \varepsilon TW(I - \varepsilon TW)^{-1}W)\mathcal{X}_D' B_D \\
& \quad \cdot ((I + TB'XL^{-1}B)^{-1} - TB_D'\mathcal{X}_D(W + \varepsilon TW(I - \varepsilon TW)^{-1}W)\mathcal{X}_D' B_D))^{-1} \\
& \quad \cdot B_D'\mathcal{X}_D(W + \varepsilon TW(I - \varepsilon TW)^{-1}W)(I + TA_f)' \\
& + \tfrac{1}{\alpha^2}G_C(I - T\tfrac{1}{\alpha^2}G'XG)^{-1}G_C' + \tfrac{1}{\alpha^2}K_D^o K_D^{o\prime} \\
& - K_D^o C_D(W^{-1} - \varepsilon TI - T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}(I + TA_f)' \\
& - (I + TA_f)(W^{-1} - \varepsilon TI - T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}C_D' K_D^{o\prime} \\
& + TK_D^o C_D(W^{-1} - \varepsilon TI - T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}C_D' K_D^{o\prime}.
\end{aligned}$$

The reliable decentralized cases can be solved similarly using the appropriate Riccati difference equations.

2.7 Theorems in Continuous-time and Difference-equation Form

The continuous-time results of Medanić et al. [26], Veillette et al. [27], [1] and Veillette [25] can be obtained from the unified conditions by taking $T = 0$. Note that then $X\Lambda^{-1} = X$ and $\Pi^{-1}Y = Y$.

Consider Theorem 2.3.3. The resulting continuous-time conditions are

$$u = K^c\xi, \qquad \dot{\xi} = A\xi + B(K^c\xi) + G(K^d\xi) + K^o(y - C\xi),$$

$$K^c = -B'X, \qquad K^d = \frac{1}{\alpha^2}G'X, \qquad K^o = (I - \frac{1}{\alpha^2}YX)^{-1}YC'$$

where $X \geq 0$ and $Y > 0$ satisfy

$$A'X + XA - X(BB' - \frac{1}{\alpha^2}GG')X + H'H + \alpha^2 C_\Omega{}' C_\Omega = 0$$

$$AY + YA' - Y(C_{\bar{\Omega}}{}' C_{\bar{\Omega}} - \frac{1}{\alpha^2}H'H)Y + GG' = 0$$

such that $\rho(YX) < \alpha^2$ and the eigenvalues of $A + GK^d - K^oC$ lie in the left half complex plane. The remaining conditions reduce to $\alpha^2 I > 0$, which is always satisfied.

The divided-difference operator discrete-time conditions are as written in the theorem with the ρ-operator replaced by the divided-difference operator.

To see the relationship between the divided-difference operator discrete form of the system and conditions and the difference-equation form, rewrite the divided-difference operator equations to express x_{k+1} and ξ_{k+1} in terms of x_k and ξ_k.

$$x_{k+1} = (I + TA)x_k + TBu_k + TGw_{0,k}$$

$$\xi_{k+1} = (I + TA)\xi_k + TBu_k + TG\hat{w}_{0,k} + TK^o(y_k - C\xi_k)$$

$$y_k = Cx_k + w_k, \qquad z_k = \begin{pmatrix} Hx_k \\ u_k \end{pmatrix}.$$

Since we would like to retain x_k and ξ_k as the state and observer variables, u_k, $w_{0,k}$, w_k and $\hat{w}_{0,k}$ as inputs, and y_k and z_k as outputs, we choose $A_q := I + TA$, $B_q := TB$, $G_q := TG$, $K^o{}_q := TK^o$, $C_q := C$, and $H_q := H$ to obtain the difference-equation form:

$$x_{k+1} = A_q x_k + B_q u_k + G_q w_{0,k}$$

$$\xi_{k+1} = A_q\xi_k + B_q u_k + G_q\hat{w}_{0,k} + K^o{}_q(y_k - C_q\xi_k)$$

$$y_k = C_q x_k + w_k, \qquad z_k = \begin{pmatrix} H_q x_k \\ u_k \end{pmatrix}.$$

We make these substitutions into the divided-difference operator form of the conditions to get the difference-equation form. First note that

$$u_k = -B'X\Lambda^{-1}(I + TA)\xi_k = -B_q'(\frac{1}{T}X)\Lambda^{-1}A_q\xi_k$$

$$\hat{w}_{0,k} = \frac{1}{\alpha^2}G'X\Lambda^{-1}(I + TA)\xi_k = \frac{1}{\alpha^2}G_q'(\frac{1}{T}X)\Lambda^{-1}A_q\xi_k$$

$$\Lambda = I + (B_qB_q' - \frac{1}{\alpha^2}G_qG_q')(\frac{1}{T}X), \qquad \Pi = I + (TY)(C_{\bar{\Omega},q}{}'C_{\bar{\Omega},q} - \frac{1}{\alpha^2}H_q'H_q)$$

where $C_{\bar{\Omega},q} := C_{\bar{\Omega}}$. A choice of X_q, Y_q, Λ_q, and Π_q that will give us a form similar in appearance to the divided-difference operator form, and which will be analogous to that given in [2], is thus $X_q := \frac{1}{T}X$, $Y_q := TY$, $\Lambda_q := \Lambda$, and $\Pi_q := \Pi$. (Note, for $T > 0$, that $X_q \geq 0$ whenever $X \geq 0$, and $Y_q > 0$ whenever $Y > 0$.) Then we have

$$K^o{}_q = TK^o = (I - \frac{1}{\alpha^2}Y_qX_q)^{-1}A_q\Pi_q{}^{-1}Y_qC_q',$$

$$\rho(Y_qX_q) < \alpha^2,$$

$$\begin{pmatrix} \alpha^2 I - G_q'\alpha^2 {Y_q}^{-1} G_q & G_q'(\alpha^2 {Y_q}^{-1} - X_q){K^o}_q \\ {K^o}_q'(\alpha^2 {Y_q}^{-1} - X_q)G_q & \alpha^2 I - {K^o}_q'(\alpha^2 {Y_q}^{-1} - X_q){K^o}_q \end{pmatrix} > 0,$$

and, for all $\omega \subseteq \Omega$,

$$\begin{pmatrix} \alpha^2 I - G_q'\alpha^2 {Y_q}^{-1} G_q & G_q'(\alpha^2 {Y_q}^{-1} - X_q){K^o}_{\bar{\omega},q} \\ {K^o}_{\bar{\omega},q}'(\alpha^2 {Y_q}^{-1} - X_q)G_q & \alpha^2 I - {K^o}_{\bar{\omega},q}'(\alpha^2 {Y_q}^{-1} - X_q){K^o}_{\bar{\omega},q} \end{pmatrix} > 0,$$

where ${K^o}_{\bar{\omega},q} := TK^o_{\bar{\omega}}$. To arrange the Riccati equations in this form, we multiply (or divide) the equations by T. Equation (2.19) can be rearranged as

$$(I + TA)\Pi^{-1}Y(I + TA)' + TGG' = Y,$$

which becomes, on multiplying by T,

$$A_q {\Pi_q}^{-1} Y_q A_q' + G_q G_q' = Y_q,$$

and Equation (2.18) can be rearranged as

$$(I + TA)'X\Lambda^{-1}(I + TA) + TH'H + T\alpha^2 C_\Omega' C_\Omega = X,$$

which becomes, on dividing by T,

$$A_q' X_q {\Lambda_q}^{-1} A_q + H_q' H_q + \alpha^2 C_{\Omega,q}' C_{\Omega,q} = X_q$$

where $C_{\Omega,q} := C_\Omega$. The stability condition $A + T\frac{1}{\alpha^2}GG'X\Lambda^{-1}(I + TA) - K^o C \in D_T$ becomes

$$\frac{1}{T}(A_q + \frac{1}{\alpha^2} G_q G_q' X_q {\Lambda_q}^{-1} A_q - {K^o}_q C_q) - \frac{1}{T} I \in D_T$$

or, equivalently, the eigenvalues of $A_q + \frac{1}{\alpha^2} G_q G_q' X_q {\Lambda_q}^{-1} A_q - {K^o}_q C_q$ lie in the unit disc. The detectability condition, (A, H) detectable becomes $(\frac{A_q - I}{T}, H_q)$ detectable, which is equivalent to (A_q, H_q) detectable.

Now consider Theorem 2.3.4. The resulting continuous-time conditions are

$$u = K^c \xi$$

$$\dot{\xi} = A\xi + B(K^c\xi) + G(K^d\xi) - B_\Omega(K^\Omega\xi) + K^o(y - C\xi)$$

$$K^c = -B'X, \qquad K^d = \frac{1}{\alpha^2}G'X, \qquad K^\Omega = -B_\Omega' X,$$

$$K^o = (I - \frac{1}{\alpha^2}YX)^{-1}YC'$$

where $X \geq 0$ and $Y > 0$ satisfy

$$A'X + XA - X(B_{\bar{\Omega}}B_{\bar{\Omega}}' - \frac{1}{\alpha^2}GG')X + H'H = 0$$

$$AY + YA' - Y(C'C - \frac{1}{\alpha^2}H'H)Y + GG' + \alpha^2 B_{\Omega}B_{\Omega}' = 0$$

such that $\rho(YX) < \alpha^2$ and the eigenvalues of $A + GK^d - K^oC + BK^c$ lie in the left half complex plane. The remaining condition $(I > 0)$ is always satisfied.

The divided-difference operator discrete-time results are as stated in the theorem with ρ replaced by δ. Thus, the observer is

$$\delta\xi = A\xi + B(K^c\xi) + G(K^d\xi) - B_{\Omega}(K^{\Omega}\xi) + K^o(y - C\xi).$$

Now consider Theorem 2.4.1. The resulting continuous-time conditions are

$$\dot{\xi}_i = A\xi_i + \sum_{j=1}^{p} B_j(K_j^c\xi_i) + G(K^d\xi_i) + K^o{}_i(y_i - C_i\xi_i)$$

$$u_i = K_i^c\xi_i, \qquad K_i^c = -B_i'X, \quad i \in \{1, 2, ..., p\}, \qquad K^d = \frac{1}{\alpha^2}G'X$$

where K_D^o block diagonal, $X \geq 0$ and $W > 0$ satisfy

$$A'X + XA - X(BB' - \frac{1}{\alpha^2}GG')X + H'H = 0$$

$$WA_e' + A_eW + WX_DB_DB_D'X_DW + \frac{1}{\alpha^2}G_CG_C' + \frac{1}{\alpha^2}K_D^oK_D^{o\prime} - K_D^oC_DW - WC_D'K_D^{o\prime} = 0$$

such that the eigenvalues of $A_e - K_D^oC_D$ lie in the left half complex plane. Here,

$$X_D := \begin{pmatrix} X & 0 & \dots & 0 \\ 0 & X & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & X \end{pmatrix}.$$

The remaining conditions reduce to $\alpha^2 I > 0$, which is always satisfied.

The divided-difference operator discrete-time conditions are as written in the theorem with the ρ-operator replaced by the divided-difference operator.

To see the relationship between the divided-difference operator discrete form of the system and conditions and the difference-equation form, rewrite the divided-difference operator equations to express x_{k+1} and $\xi_{i,k+1}$ in terms of x_k and $\xi_{i,k}$.

$$x_{k+1} = (I + TA)x_k + \sum_{i=1}^{p} TB_iu_{i,k} + TGw_{0,k}, \quad x_0 = 0,$$

$$\xi_{i,k+1} = (I + TA)\xi_{i,k} + \sum_{j=1}^{p} TB_j \hat{u}^i_{j,k} + TG\hat{w}^i_{0,k} + TK^o{}_i(y_{i,k} - C_i\xi_{i,k})$$

$$y_{i,k} = C_i x_k + w_{i,k}, \qquad z_k = \begin{pmatrix} Hx_k \\ u_{1,k} \\ \vdots \\ u_{p,k} \end{pmatrix}.$$

Since we would like to retain x_k and $\xi_{i,k}$ as the state and observer variables, $u_{i,k}$, $w_{0,k}$, $w_{i,k}$, $\hat{u}^i_{j,k}$ and $\hat{w}^i_{0,k}$ as inputs, and $y_{i,k}$ and z_k as outputs, we choose $A_q := I + TA$, $B_{i,q} := TB_i$, $G_q := TG$, $K^o{}_{i,q} := TK^o{}_i$, $C_{i,q} := C_i$, and $H_q := H$ to obtain the difference-equation form:

$$x_{k+1} = A_q x_k + \sum_{i=1}^{p} B_{i,q} u_{i,k} + G_q w_{0,k}, \quad x_0 = 0,$$

$$\xi_{i,k+1} = A_q \xi_{i,k} + \sum_{j=1}^{p} B_{j,q} \hat{u}^i_{j,k} + G_q \hat{w}^i_{0,k} + K^o{}_{i,q}(y_{i,k} - C_{i,q}\xi_{i,k})$$

$$y_{i,k} = C_{i,q} x_k + w_{i,k}, \qquad z_k = \begin{pmatrix} H_q x_k \\ u_{1,k} \\ \vdots \\ u_{p,k} \end{pmatrix}.$$

We make these substitutions into the divided-difference operator form of the conditions to find the difference-equation form. First note that

$$u_{i,k} = -B_i' X \Lambda^{-1}(I + TA)\xi_{i,k} = -B_{i,q}'(\frac{1}{T}X)\Lambda^{-1}A_q\xi_{i,k}$$

$$\hat{w}^i_{0,k} = \frac{1}{\alpha^2}G' X \Lambda^{-1}(I + TA)\xi_{i,k} = \frac{1}{\alpha^2}G_q'(\frac{1}{T}X)\Lambda^{-1}A_q\xi_{i,k}$$

$$\Lambda = I + (\sum_{i=1}^{p} B_{i,q}B_{i,q}' - \frac{1}{\alpha^2}G_q G_q')(\frac{1}{T}X).$$

A choice of X_q and Λ_q that will give us a form similar in appearance to the divided-difference operator form, and which will be analogous to that given in [2], is thus $X_q := \frac{1}{T}X$ and $\Lambda_q := \Lambda$. (Note, for $T > 0$, that $X_q \geq 0$ whenever $X \geq 0$.) Also, since

$$K^o{}_{i,q} = TK^o{}_i = \alpha^2(TW_i)C_i',$$

define $W_q := TW$. Then we have

$$\Xi_D(X_q, W_q) = \begin{pmatrix} \alpha^2 I - G_q' X_q G_q - G_{C,q}' W_q^{-1} G_{C,q} & G_{C,q}' W_q^{-1} K^o{}_{D,q} \\ K^{o\prime}{}_{D,q} W_q^{-1} G_{C,q} & \alpha^2 I - K^{o\prime}{}_{D,q} W_q^{-1} K^o{}_{D,q} \end{pmatrix} > 0.$$

To arrange the design equations in this form, we divide (or multiply) the equations by T. Thus,

$$(I+TA)'X\Lambda^{-1}(I+TA)+TH'H=X$$

becomes, on dividing by T,

$$A_q'X_q\Lambda_q{}^{-1}A_q+H_q'H_q=X_q,$$

and (A.12) becomes, on multiplying by T,

$$[A_{e,q}-K^o{}_{D,q}C_{D,q}](W_q^{-1}-\mathcal{X}_{D,q}'B_{D,q}(I+B_q'X_qL_q^{-1}B_q)B_{D,q}'\mathcal{X}_{D,q})^{-1}[A_{e,q}-K^o{}_{D,q}C_{D,q}]'$$
$$+\tfrac{1}{\alpha^2}G_{C,q}(I-\tfrac{1}{\alpha^2}G_q'X_qG_q)^{-1}G_{C,q}'+\tfrac{1}{\alpha^2}K^o{}_{D,q}K^{o\prime}{}_{D,q}=W_q,$$

where $A_{e,q}$ and the other matrices are defined in an appropriate manner, substituting for all matrices in terms of their difference-equation counterparts.

The stability condition $A_e-K_D^oC_D\in D_T$ becomes

$$\frac{1}{T}(A_{e,q}-K^o{}_{D,q}C_{D,q})-\frac{1}{T}I\in D_T$$

or, equivalently, the eigenvalues of $A_{e,q}-K^o{}_{D,q}C_{D,q}$ lie in the unit disc. The detectability condition, (A,H) detectable becomes $(\frac{A_q-I}{T},H_q)$ detectable, which is equivalent to the condition that (A_q,H_q) be detectable.

2.8 Example

Consider the following example:

Example 2.8.1

$$\dot{x}=\begin{pmatrix}-2&1&1&1\\3&0&0&2\\-1&0&-2&-3\\-2&-1&2&-1\end{pmatrix}x+\begin{pmatrix}0\\1\\0\\0\end{pmatrix}u_1+\begin{pmatrix}0\\0\\0\\1\end{pmatrix}u_2+\begin{pmatrix}1\\0\\1\\0\end{pmatrix}w_0$$

$$y_1=(1\ 0\ 0\ 0)+w_1$$

$$y_2=(0\ 0\ 1\ 0)+w_2$$

$$z=\begin{pmatrix}(1\ 0\ -1\ 0)x\\u_1\\u_2\end{pmatrix}.$$

First, a comparison of the performance of the basic centralized and decentralized controller designs is given in Table 2.1 for the discretized system with sampling interval $T = 0.1$.

If the basic decentralized design is implemented, then stability is lost when sensor y_1 fails. A reliable decentralized controller was found, reliable to failures in sensor y_1, using the conservative design norm bound $\alpha = 23$. With this controller, the actual closed-loop norm was less than 7 despite failures of either sensor. This was determined by calculating the H_∞ norm of the closed-loop system by finding the smallest value α for which the Hamiltonian matrix had no eigenvalues on the boundary of the stability region. Table 2.2 shows how the reliable decentralized controller design compares with the basic decentralized design when either sensor fails.

For the reliable design, we chose the sampling period, $T = 0.01$, to be about 1/20th of the unstable plant pole value of 0.19. The basic controller designs were found for slow sampling of $T = 0.1$, but convergence becomes a problem for the reliable design with the minimum design α growing rapidly as T increases.

Table 2.1: Closed-loop spectra and H_∞ norms for varying α.
Discrete-time, T=0.1 (Spectra in z-plane)

	Centralized Observer Controller		Decentralized Controller	
	Spectrum	$\|T\|_\infty$	Spectrum	$\|T\|_\infty$
$\alpha = 20$	$0.83 \pm j0.25$ $0.84 \pm j0.25$ 0.98 0.94 0.78 0.76	3.17	0.98 $0.82 \pm j0.25$ 0.96 $0.83 \pm j0.25$ 0.89 $0.85 \pm j0.25$ 0.78 0.78 0.76	3.64
$\alpha = 16$	$0.83 \pm j0.25$ $0.84 \pm j0.25$ 0.98 0.94 0.78 0.76	3.16	0.98 $0.82 \pm j0.25$ 0.96 $0.83 \pm j0.25$ 0.89 $0.85 \pm j0.25$ 0.78 0.78 0.76	3.63
$\alpha = 12$	$0.83 \pm j0.25$ $0.84 \pm j0.25$ 0.98 0.94 0.78 0.76	3.13	0.98 $0.82 \pm j0.25$ 0.96 $0.83 \pm j0.25$ 0.89 $0.85 \pm j0.25$ 0.78 0.78 0.76	3.59
$\alpha = 8$	$0.83 \pm j0.25$ $0.84 \pm j0.25$ 0.98 0.94 0.78 0.76	3.07	0.97 $0.82 \pm j0.25$ 0.96 $0.83 \pm j0.25$ 0.89 $0.85 \pm j0.25$ 0.78 0.78 0.76	3.49
$\alpha = 4$	$0.83 \pm j0.25$ $0.84 \pm j0.25$ 0.97 0.94 0.78 0.76	2.77	0.88 $0.82 \pm j0.25$ 0.78 $0.83 \pm j0.25$ 0.78 $0.85 \pm j0.25$ 0.76 $0.96 \pm j0.01$	3.05
$\alpha = 2$	$0.83 \pm j0.25$ $0.84 \pm j0.25$ 0.93 0.90 0.78 0.75	1.959	0.95 $0.82 \pm j0.25$ 0.85 $0.83 \pm j0.25$ 0.78 $0.85 \pm j0.26$ 0.74 $0.76 \pm j0.09$	1.994
$\alpha = 1.7$	$0.82 \pm j0.25$ $0.84 \pm j0.25$ $0.77 \pm j0.003$ 0.94 0.59	1.696	none	none

Table 2.2: H_∞ norms for basic and reliable decentralized designs.
Discrete-time, $T = 0.01$

	Basic Controller				Reliable Controller (to $y_1 = 0$)			
	ε	no failure	$y_1 = 0$	$y_2 = 0$	ε	no failure	$y_1 = 0$	$y_2 = 0$
$\alpha = 29$	0.006	3.57	unstable	5.17	0.0001	6.34	5.96	6.35
$\alpha = 26$	0.005	3.58	unstable	5.19	0.0001	6.52	6.04	6.54
$\alpha = 23$	0.003	3.60	unstable	5.25	0.0001	6.76	6.13	6.80
$\alpha = 1.83$	0.0001	1.8291	unstable	2.22	No solution found.			

CHAPTER 3

SAMPLED-DATA CONTROLLER DESIGN FOR CONTINUOUS-TIME NORM BOUNDING

3.1 Introduction

Traditionally, digital controllers have been designed for continuous-time plants either by designing an appropriate continuous-time controller and discretizing it, or by discretizing the plant and designing a discrete-time controller for the resulting discrete-time system. The first approach neglects the resulting errors in the implementation, which may compound over time. The second approach guarantees performance at the sampling times but disregards performance between sampling times.

It is generally supposed that, if sampling is "fast enough," either of these approaches will result in satisfactory performance. However, fast sampling is not always convenient since many actuators and sensors have limited time-response characteristics.

Recent breakthroughs yield sampled-data controller designs that optimize closed-loop system behavior over all time, not just at sampling instances. We describe some of those results and then present a new approach, which we believe provides more insight into the development of these new design methods as well as demonstrating which signal norm should be considered for the mixed continuous and piecewise-constant signal space.

Başar [9] derives the optimal time-varying controller design that minimizes the H_∞ norm of a continuous-time plant with sampled measurements, using a game-theoretic approach. Toivonen [10] treats the H_∞-optimal finite-horizon control problem for time-varying continuous-time systems with sampled-data controllers, piecewise constant on each sampling interval, using a game-theoretic approach, resulting in a series of Riccati equations that must be solved in reverse time.

Bamieh and Pearson [12] and Toivonen [11] also treat the H_∞-optimal control problem for continuous-time systems with sampled-data controllers, piecewise constant on each sampling interval. They treat it in the frequency domain by using a lifting of the continuous-time signals into an infinite-dimensional signal space and then by showing that the part of the signal of interest is actually finite-dimensional. Their infinite-dimensional problem then reduces to a finite-dimensional discrete-time H_∞ problem of a different form from that of the original problem.

We take a new look at the infinite-horizon, time-invariant problem using a different state-space method, resulting in a single closed-form design equation for the H_∞-norm-bounding state-feedback controller, by taking the limit as a parameter tends to infinity. Taking the same approach, we derive design equations for decentralized H_∞-norm-bounding controllers for the same sampled-data problem.

The sampled-data design problem is posed as the limit of a sequence of two-rate digital control design problems in the following sense. The desired sampling rate $1/T$ is used for the control variable. However, to measure continuous-time performance, the disturbance and regulated-output variables are uniformly sampled N times in each sampling interval T of the controller. The norm of the two-rate problems is selected so that, when the limit is taken of this problem as $N \to \infty$, the continuous-time norm is recovered. The two-rate problem is expressed using the divided-difference operator approach of Middleton and Goodwin [5], extended to multirate systems (see also [28]). This is done to make the limiting process more transparent. The lifting technique of Meyer and Burrus [13], Buescher and Grizzle [14], [15], and D. G. Meyer [29], [16], is then used to find a single-rate form of the two-rate system. A new bounded real lemma is developed, an extension of the bounded real lemma in [30], that guarantees stability and an H_∞-norm bound if certain conditions are satisfied. This is applied to the closed-loop system equations, yielding design equations for the controller gain in the two-rate case. The limit is then taken as the sampling period on the disturbance and performance variable tends to zero. Closed-form design equations for the controller are thus found. A proof is then given that this limiting controller guarantees stability and an H_∞-norm bound for the limiting closed-loop continuous-time system. A restriction to the method is that the class of disturbances considered is assumed to be Riemann integrable as well as in $\mathcal{L}_2$.

For technical reasons in the proof, the controller designs for the two-rate problems must guarantee a degree of stability for the resulting closed-loop systems so that the limiting controller will stabilize the closed-loop system. Modifications to the theorems that guarantee a degree of stability for the closed-loop system are indicated. The definition chosen for the degree of stability of a sampled-data system is chosen to be Definition 2.5.1, which is based on the degree of stability of the underlying continuous-time system, so that the limiting behavior will make sense.

In Section 3.2, the problem setup is given, and the appropriate two-rate problems are defined and transformed into single-rate forms. In Section 3.3, the required bounded real lemma is presented, and the design equations for the two-rate state-feedback controller gains and the two-rate decentralized controller gains are given. In Section 3.4, the limit is taken, and the desired design equations are found for each case. A proof is then given that the limiting controllers satisfy the performance criteria in the original sampled-data problems. In Section 3.5, the sensor-outage reliable decentralized controller design is found. In Section 3.6, numerical methods for the solution of the design equations are discussed. In Section 3.7, an example is presented for comparison of the decentralized sampled-data controller design to that of the discrete-time controller design of Section 2.4. A performance measure is derived to compare the performance of the two controllers. The proof of the bounded real lemma and the derivations of the two-rate design equations are relegated to Appendix B since they are not central to the development.

3.2 Problem Formulation

We first discuss the problem setup and related two-rate problem for the state-feedback control problem and then generalize the approach to the decentralized controller design problem.

3.2.1 State-feedback problem

Problem 3.2.1 *Given the continuous-time plant*

$$\dot{x} = A_c x + B_c u + G_c w_0, \quad x(0) = 0, \tag{3.1}$$

with the performance-output variable

$$z = \begin{pmatrix} H_c x \\ u \end{pmatrix}, \tag{3.2}$$

design a linear time-invariant zero-order hold controller of the state-feedback form

$$u(t) = K^c x(kT), \quad t \in [kT, (k+1)T); \qquad k = 0, 1, 2, \ldots, \tag{3.3}$$

for which the closed-loop continuous-time system is exponentially stable and has H_∞ norm from w_0 to z less than a given value α.

As a result of the approach taken in this thesis, the problem solved herein bounds the following norm:

Definition 3.2.1 *The* **Riemann H_∞ norm** *of a linear system with input v and output z is defined to be*

$$\sup_{v \in \mathcal{L}_2 \cap \mathcal{RI}} \frac{\|z\|_2}{\|v\|_2},$$

where $v \in \mathcal{RI}$ if v is Riemann integrable and $v \in \mathcal{L}_2$ if

$$\|v(t)\|_2 := \left(\int_0^\infty v(t)'v(t)\,dt\right)^{\frac{1}{2}} < \infty.$$

This is different from the standard definition, which involves the supremum over all Lebesgue-integrable signals.

To develop such a design method, we first develop a design method for controllers of the form (3.3) that guarantee closed-loop stability and the H_∞-norm bound α for the closed-loop synchronously sampled two-rate discrete-time system where $w_0(t)$ and $z(t)$ are sampled every $\frac{T}{N}$ time units. By sampling the disturbance and regulated output between control sampling instances, performance is regulated between sampling instances. As $N \to \infty$, a bound on the continuous-time performance measure is obtained. We utilize the divided-difference equation form of Middleton and Goodwin [5], generalized to multirate systems. The discrete-time norm must also be chosen appropriately to agree with the continuous-time norm in the limit as $N \to \infty$. As noted by Chen and Francis [18], not all sampled $\mathcal{L}_2$ signals are in ℓ_2, and the discrete-time H_∞ norm is defined for disturbances in ℓ_2. However, Riemann-integrable signals in $\mathcal{L}_2$ are shown to be the limit as the sampling interval approaches zero of sampled sequences in ℓ_2.

First, we find the divided-difference form for the two-rate system. Suppose that the control $u(t)$ is constant on each time interval $[kT, (k+1)T)$, $k = 0, 1, \ldots$, the disturbance $w_0(t)$ is constant on each interval $[i\frac{T}{N}, (i+1)\frac{T}{N})$, and the performance variable $z(t)$ is sampled at $i\frac{T}{N}$, $i = 0, 1, \ldots$. Then the overall two-rate system is periodic with period T. Thus, we require the update equation for $x((k+1)T)$ in terms of $x(kT)$:

$$\begin{aligned} x((k+1)T) &= \left(e^{A_cT}\right)x(kT) + \left(\int_0^T e^{A_ct}\,dt\,B_c\right)u(kT) \\ &\quad + \left(e^{A_c(\frac{N-1}{N})T}\int_0^{\frac{T}{N}} e^{A_ct}\,dt\;Gc \;\;\ldots\;\; \int_0^{\frac{T}{N}} e^{A_ct}\,dtG_c\right)w^0(kT), \end{aligned}$$

where

$$w^0(kT) := \begin{pmatrix} w_0(kT) \\ \vdots \\ w_0(kT + (\frac{N-1}{N})T) \end{pmatrix}.$$

This can be rewritten in divided difference form to obtain

$$\begin{aligned} \delta x &:= \tfrac{x((k+1)T) - x(kT)}{T} \\ &= \left(\tfrac{e^{A_cT} - I}{T}\right)x(kT) + \left(\tfrac{1}{T}\int_0^T e^{A_ct}\,dtB_c\right)u(kT) \\ &\quad + \left(e^{A_c(\frac{N-1}{N})T}\tfrac{1}{T}\int_0^{\frac{T}{N}} e^{A_ct}\,dtG_c \;\;\ldots\;\; \tfrac{1}{T}\int_0^{\frac{T}{N}} e^{A_ct}\,dtG_c\right)w^0(kT), \\ &=: A_\delta x(kT) + B_\delta u(kT) + G_\delta w^0(kT). \end{aligned}$$

Notice that, as $N \to \infty$, this expression reduces to

$$\delta x = A_\delta x + B_\delta u + \frac{1}{T}\int_{kT}^{(k+1)T} e^{A_c((k+1)T-t)} G_c w_0(t)\, dt,$$

and, in the limit as $T \to 0$, this reduces to the continuous-time equation.

Compare the Riemann H_∞ norm of the continuous-time system with the H_∞ norm of a single-rate discrete-time system:

$$\sup_{w_0 \in \mathcal{L}_2 \cap \mathcal{RI}} \frac{\int_0^\infty z(t)'z(t)\, dt}{\int_0^\infty w_0(t)'w_0(t)\, dt}; \qquad \sup_{w_0 \in \ell_2} \frac{\sum_{j=0}^\infty z(jT)'z(jT)}{\sum_{j=0}^\infty w_0(jT)'w_0(jT)}.$$

Rewriting the discrete-time H_∞ norm by multiplying the numerator and denominator by T, we obtain

$$\sup_{w_0 \in \ell_2} \frac{T\sum_{j=0}^\infty z(jT)'z(jT)}{T\sum_{j=0}^\infty w_0(jT)'w_0(jT)}.$$

Consider

$$\frac{T\sum_{j=0}^\infty z(jT)'z(jT)}{T\sum_{j=0}^\infty w_0(jT)'w_0(jT)}.$$

In the limit as $T \to 0$, this converges, if $z(t)$ and $w_0(t)$ are Riemann integrable, to

$$\frac{\int_0^\infty z(t)'z(t)\, dt}{\int_0^\infty w_0(t)'w_0(t)\, dt}.$$

We choose the discrete-time norm based on this observation so that the ratio approximates the integral and converges to the integral as $N \to \infty$. However, in the discrete-time norm, the supremum is taken over the set of ℓ_2 disturbances and we would like to bound the supremum over $\mathcal{L}_2$ signals. Not all sampled $\mathcal{L}_2$ signals are in ℓ_2, but a Riemann-integrable $\mathcal{L}_2$ signal is the limit, as $N \to \infty$, of ℓ_2 signals, as shown next.

Suppose the sequence $\{w_0(jT)\}_{j=0,1,\ldots}$ is obtained by sampling a Riemann-integrable $\mathcal{L}_2$ signal $w_0(t)$. Then there exists $c > 0$ such that

$$\int_0^\infty w(t)'w(t)\, dt < c.$$

Since $w(t)'w(t)$ is Riemann integrable, the Riemann sums

$$\frac{T}{N}\sum_{i=0}^\infty w(i\frac{T}{N})'w(i\frac{T}{N}) \longrightarrow \int_0^\infty w(t)'w(t)\, dt$$

as $N \to \infty$. Let $C > c$. Then there exists $N_1 > 0$, such that, for all $N > N_1$,

$$\frac{T}{N}\sum_{i=0}^\infty w(i\frac{T}{N})'w(i\frac{T}{N}) < C.$$

Thus, for all $N > N_1$,

$$\left\{ w(i\frac{T}{N}) \right\}_{i=0,1,\ldots} \in \ell_2.$$

Thus, for N sufficiently large, the resulting sampled sequences are in ℓ_2.

Now we find the ratio for the norm in the two-rate problems. As $N \to \infty$,

$$\frac{\frac{T}{N}\sum_{i=0}^{\infty} z(i\frac{T}{N})'z(i\frac{T}{N})}{\frac{T}{N}\sum_{i=0}^{\infty} w_0(i\frac{T}{N})'w_0(i\frac{T}{N})} \longrightarrow \frac{\int_0^{\infty} z(t)'z(t)\,dt}{\int_0^{\infty} w_0(t)'w_0(t)\,dt},$$

even though $z(t)$ is not continuous in time. But also

$$\frac{\frac{T}{N}\sum_{i=0}^{\infty} z(i\frac{T}{N})'z(i\frac{T}{N})}{\frac{T}{N}\sum_{i=0}^{\infty} w_0(i\frac{T}{N})'w_0(i\frac{T}{N})} = \frac{\frac{T}{N}\sum_{i=0}^{\infty} x(i\frac{T}{N})'H_c'H_c x(i\frac{T}{N}) + T\sum_{k=0}^{\infty} u(kT)'u(kT)}{\frac{T}{N}\sum_{k=0}^{\infty} w^0(kT)'w^0(kT)}$$

since $u(t)$ is piecewise constant on $t \in [kT,(k+1)T)$, $k = 0,1,\ldots$. (Note that, as $N \to \infty$, this approaches

$$\frac{\int_0^{\infty} x(t)'H_c'H_c x(t)\,dt + T\sum_{k=0}^{\infty} u(kT)'u(kT)}{\int_0^{\infty} w_0(t)'w_0(t)\,dt}.)$$

To express the multirate problem as a single-rate problem, we need the following expression for $H_c x(i\frac{T}{N})$, $i = 0,1,\ldots$, in terms of $x(kT)$, $u(kT)$, and $w^0(kT)$:

$$\begin{pmatrix} H_c x(kT) \\ H_c x(kT + (\frac{1}{N})T) \\ \vdots \\ H_c x(kT + (\frac{N-1}{N})T) \end{pmatrix} = \begin{pmatrix} H_c \\ H_c e^{A_c(\frac{1}{N})T} \\ \vdots \\ H_c e^{A_c(\frac{N-1}{N})T} \end{pmatrix} x(kT) + \begin{pmatrix} 0 \\ H_c \int_0^{(\frac{1}{N})T} e^{A_c t}\,dt\,B_c \\ \vdots \\ H_c \int_0^{(\frac{N-1}{N})T} e^{A_c t}\,dt\,B_c \end{pmatrix} u(kT)$$

$$+ \begin{pmatrix} 0 & \cdots & 0 \\ H_c \int_0^{\frac{T}{N}} e^{A_c t}\,dt\,G_c & \ddots & \vdots \\ \vdots & \ddots & \\ H_c e^{A_c(\frac{N-2}{N})T} \int_0^{\frac{T}{N}} e^{A_c t}\,dt\,G_c \quad \cdots & H_c \int_0^{\frac{T}{N}} e^{A_c t}\,dt\,G_c & 0 \end{pmatrix} w^0(kT)$$

$$=: H_\delta x(kT) + D_{BH}u(kT) + D_{GH}w^0(kT).$$

Then,

$$\frac{T}{N}\sum_{i=0}^{\infty} x(i\tfrac{T}{N})'H_c'H_c x(i\tfrac{T}{N})$$
$$= \frac{T}{N}\sum_{k=0}^{\infty}(H_\delta x(kT) + D_{BH}u(kT) + D_{GH}w^0(kT))'(H_\delta x(kT) + D_{BH}u(kT) + D_{GH}w^0(kT)).$$

Then, the H_∞ norm can be expanded as

$$\sup_{w^0 \in \ell_2} \frac{T\sum_{k=0}^{\infty}\tilde{z}(kT)'\tilde{z}(kT)}{T\sum_{k=0}^{\infty}\tilde{w}(kT)'\tilde{w}(kT)},$$

where

$$\tilde{w}(kT) := \frac{1}{\sqrt{N}} w^0(kT)$$

and

$$\begin{aligned}\tilde{z}(kT) \quad &:= \quad \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta x(kT) + D_{BH}u(kT) + D_{GH}w^0(kT)) \\ u(kT) \end{pmatrix} \\ &= \quad \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta x(kT) + D_{BH}u(kT)) + D_{GH}\tilde{w}(kT) \\ u(kT) \end{pmatrix}.\end{aligned}$$

We have the following single-rate formulation for the two-rate problem.

Problem 3.2.2 *Given the discrete-time plant*

$$\delta x = A_\delta x + B_\delta u + (\sqrt{N}G_\delta)\tilde{w}, \quad x(0) = 0, \tag{3.4}$$

with the performance-output variable

$$\tilde{z} = \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta x + D_{BH}u) + D_{GH}\tilde{w} \\ u \end{pmatrix}, \tag{3.5}$$

design a linear time-invariant controller of the state-feedback form $u = K^c x$, for which the closed-loop system is stable and has H_∞ norm from $\tilde{w}$ to $\tilde{z}$ less than a given value α.

3.2.2 Decentralized control problem

Consider now the decentralized continuous-time plant

$$\dot{x} = A_c x + \sum_{i=1}^{p} B_c^i u_i + G_c w_0, \qquad x(0) = 0$$

where u_i is the control input at the ith subsystem and w_0 is an unknown disturbance. Suppose further that only the measurements

$$y_i(kT) = C_i x(kT) + w_i(kT), \qquad k = 0, 1, \ldots,$$

are available at the ith subsystem, where $w_i(kT)$ is the measurement noise at the ith subsystem's sensors at time kT. The controller at each subsystem may depend only on the measurement and input information at that subsystem.

We would like to design a sampled-data decentralized controller that guarantees closed-loop stability and a prespecified H_∞-norm bound for the continuous-time closed-loop system from the disturbance and noise inputs

$$w_e := \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_p \end{pmatrix}$$

to the performance-output variable

$$z = \begin{pmatrix} H_c x \\ u_1 \\ \vdots \\ u_p \end{pmatrix}.$$

Such designs were presented for continuous-time decentralized controllers by Veillette, Medanić, and Perkins in [1]. Similar sampled-data decentralized controller designs were presented in Chapter 2; however, those designs guarantee a bound on the H_∞ norm of the corresponding discrete-time system, not of the continuous-time system. We extend those results using the approach of this chapter to the sampled-data problem with the continuous-time H_∞-norm bound.

A specific form is chosen for the full-order observers to match the divided-difference form of the sampled-data decentralized plant. The divided-difference form update for the sampled-data decentralized plant, with samples taken at $t = kT$, $k = 0, 1, ..., $ is

$$\delta x = A_\delta x(kT) + \sum_{j=1}^{p} B_\delta^j u_j(kT) + \frac{1}{T}\int_{kT}^{(k+1)T} e^{A_c((k+1)T-t)} G_c w_0(t)\, dt, \qquad x(0) = 0,$$

where A_δ is defined as before and B_δ^j, $j = 1, ..., p$, is defined as

$$B_\delta^j := \frac{1}{T}\int_0^T e^{A_c t}\, dt\, B_c^j.$$

The sampled-data controller is of the form

$$u_i(t) = K_i^c \xi_i(kT), \quad t \in [kT, (k+1)T); \qquad k = 0, 1, 2, ...,$$

for $i = 1, 2, ..., p$, where the observer at the ith subsystem is chosen to have the form

$$\delta\xi_i = A_\delta \xi_i(kT) + B_\delta^i u_i(kT) + \sum_{j \neq i} B_\delta^j \hat{u}_j^i(kT) + \widehat{Gw}_0^i(kT) + K_i^o(y_i(kT) - C_i\xi_i(kT)), \quad \xi_i(0) = 0.$$

Let the estimates for the other subsystems' control inputs and the disturbance, $\hat{u}_j^i(kT)$ and $\widehat{Gw}_0^i(kT)$, be based on the local subsystem's state estimate as follows:

$$\hat{u}_j^i(kT) := K_j^c \xi_i(kT), \qquad \widehat{Gw}_0^i(kT) := \widehat{GK}^d \xi_i(kT).$$

Now, we can formally state the sampled-data decentralized controller design problem.

Problem 3.2.3 *Given the decentralized continuous-time plant*

$$\dot{x} = A_c x + \sum_{i=1}^{p} B_c^i u_i + G_c w_0, \qquad x(0) = 0 \tag{3.6}$$

$$y_i(kT) = C_i x(kT) + w_i(kT), \qquad k = 0, 1, ..., \quad i \in \{1, 2, ..., p\}, \tag{3.7}$$

with the performance-output variable

$$z = \begin{pmatrix} H_c x \\ u_1 \\ \vdots \\ u_p \end{pmatrix}, \tag{3.8}$$

design a decentralized discrete-time controller of the form

$$u_i(t) = K_i^c \xi_i(kT), \quad t \in [kT, (k+1)T); \qquad k = 0, 1, 2, ..., \tag{3.9}$$

for $i = 1, 2, ..., p$, where $\xi_i(kT)$ is the state of the full-state observer at the ith subsystem

$$\delta\xi_i = \left(A_\delta + B_\delta^i K_i^c + \sum_{j \neq i} B_\delta^j K_j^c + \widehat{GK^d} \right) \xi_i(kT) + K_i^o(y_i(kT) - C_i \xi_i(kT)), \quad \xi_i(0) = 0,$$

at time kT, for which the closed-loop continuous-time system is exponentially stable and has H_∞ norm from w_e to z less than a given value α.

Since $y_i(t)$ is sampled only at times $t = kT$, $k = 0, 1, ...,$ only the values of the noise $w_i(t)$ at $t = kT$, $k = 0, 1, ...,$ affect the system. For purposes of computing the continuous-time H_∞ norm of the system, we assume that $w_i(t)$ is piecewise constant on the intervals $[kT, (k+1)T)$, $k = 0, 1, ...$.

Consider the synchronously sampled two-rate discrete-time system where $w_0(t)$, $w_1(t)$,...,$w_p(t)$, and $z(t)$ are sampled every $\frac{T}{N}$ time units. The state equation for this two-rate system can be written as

$$\delta x = A_\delta x(kT) + \sum_{i=1}^{p} B_\delta^i u_i(kT) + G_\delta w^0(kT),$$

where G_δ and $w^0(kT)$ are defined as before. The H_∞ norm is then

$$\sup_{w_0 \in \ell_2} \frac{\frac{T}{N} \sum_{i=0}^{\infty} x(i\frac{T}{N})' H_c' H_c x(i\frac{T}{N}) + T \sum_{l=1}^{p} \sum_{k=0}^{\infty} u_l(kT)' u_l(kT)}{\frac{T}{N} \sum_{k=0}^{\infty} w^0(kT)' w^0(kT) + T \sum_{l=1}^{p} \sum_{k=0}^{\infty} w_l(kT)' w_l(kT)}.$$

Again, expanding $H_c x(i\frac{T}{N})$, $i = 0, 1, ...,$ in terms of $x(kT)$, $u_i(kT)$, $i = 1, ..., p$, and $w^0(kT)$, we obtain

$$\begin{pmatrix} H_c x(kT) \\ H_c x(kT + (\frac{1}{N})T) \\ \vdots \\ H_c x(kT + (\frac{N-1}{N})T) \end{pmatrix} = H_\delta x(kT) + \sum_{i=1}^{p} D_{BH}^i u_i(kT) + D_{GH} w^0(kT),$$

where

$$D_{BH}^i := \begin{pmatrix} 0 \\ H_c \int_0^{(\frac{1}{N})T} e^{A_c t}\, dt\, B_c^i \\ \vdots \\ H_c \int_0^{(\frac{N-1}{N})T} e^{A_c t}\, dt\, B_c^i \end{pmatrix}$$

and H_δ and D_{GH} are defined as before. Then, the H_∞ norm can be expanded as

$$\sup_{w^0 \in \ell_2} \frac{T \sum_{k=0}^{\infty} \tilde{z}(kT)'\tilde{z}(kT)}{T \sum_{k=0}^{\infty} \begin{pmatrix} \tilde{w}(kT) \\ w_1(kT) \\ \vdots \\ w_p(kT) \end{pmatrix}' \begin{pmatrix} \tilde{w}(kT) \\ w_1(kT) \\ \vdots \\ w_p(kT) \end{pmatrix}},$$

where

$$\tilde{w}(kT) := \frac{1}{\sqrt{N}} w^0(kT)$$

and

$$\tilde{z}(kT) := \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta x(kT) + \sum_{i=1}^{p} D_{BH}^i u_i(kT)) + D_{GH}\tilde{w}(kT) \\ u_1(kT) \\ \vdots \\ u_p(kT) \end{pmatrix}.$$

The observer form for two-rate system can be written as

$$\delta\xi_i = A_\delta \xi_i(kT) + B_\delta^i u_i(kT) + \sum_{j \neq i} B_\delta^j \hat{u}_j^i(kT) + (\sqrt{N} G_\delta)\hat{w}^i(kT) + K_i^o(y_i(kT) - C_i \xi_i(kT)), \quad \xi_i(0) = 0,$$

$$\hat{w}^i(kT) := K^d \xi_i(kT),$$

where a specific form is now given for the estimate $\widehat{Gw_0}^i(kT)$.

Then we have the following single-rate formulation for the two-rate problem.

Problem 3.2.4 *Given the discrete-time decentralized plant*

$$\delta x = A_\delta x + \sum_{i=1}^{p} B_\delta^i u_i + (\sqrt{N} G_\delta)\tilde{w}, \quad x(0) = 0, \tag{3.10}$$

$$y_i = C_i x + w_i, \quad k = 0, 1, \ldots \quad i \in \{1, 2, \ldots, p\} \tag{3.11}$$

with the performance-output variable

$$\tilde{z} = \begin{pmatrix} \frac{1}{\sqrt{N}}\left(H_\delta x + \sum_{j=1}^{p} D_{BH}^j u_j\right) + D_{GH}\tilde{w} \\ u_1 \\ \vdots \\ u_p \end{pmatrix}, \tag{3.12}$$

design a decentralized discrete-time controller of the form $u_i = K_i^c \xi_i$, *where*

$$\delta \xi_i = \left(A_\delta + B_\delta^i K_i^c + \sum_{j \neq i} B_\delta^j K_j^c + (\sqrt{N} G_\delta) K^d \right) \xi_i + K_i^o (y_i - C_i \xi_i), \quad \xi_i(0) = 0, \tag{3.13}$$

for which the closed-loop system is stable and has H_∞ *norm from*

$$\begin{pmatrix} \tilde{w} \\ w_1 \\ \vdots \\ w_p \end{pmatrix}$$

to $\tilde{z}$ *less than a given value* α.

3.3 Solutions to Two-rate Problems

The two-rate problems, Problems 3.2.2 and 3.2.4, are solved by applying the conditions in the following generalized bounded real lemma to the closed-loop system equations and then finding formulas for the controller gains that satisfy those conditions.

The extension to the bounded real lemma derived in [30] is as follows.

Lemma 3.3.1 *Consider a linear system* T_{wz} *with a detectable realization*

$$\rho x = Fx + Gw, \qquad x(0) = 0,$$

$$z = Hx + Ew.$$

If there exist a real, symmetric matrix $X \geq 0$ *and a real* $\alpha > 0$ *such that*

(i) $F'X + XF + TF'XF + H'H$
$+ (H'E + (I + TF)'XG)(\alpha^2 I - TG'XG - E'E)^{-1}(E'H + G'X(I + TF)) \leq 0$

(ii) $\alpha^2 I - TG'XG - E'E > 0$,

then

(a) the eigenvalues of F *lie in* D_T, *the stability region for sampling interval* T

(b) $\|T_{wz}\|_\infty \leq \alpha$.

The stability region for the divided-difference form of discrete-time equations with sampling period T is a disk of radius $\frac{1}{T}$ centered at $-\frac{1}{T}$ in the γ-transform plane (see Middleton and Goodwin [5]).

Proof See Appendix B.1. □

If, in addition, we require the system to have degree of stability η, as defined in Definition 2.5.1, the generalized bounded real lemma is modified as follows:

Lemma 3.3.2 *Consider a linear system T_{wz} with a realization*

$$\rho x = Fx + Gw, \qquad x(0) = 0,$$

$$z = Hx + Ew$$

with all unobservable modes of (F, H) in D_T^η. If there exist a real, symmetric matrix $X \geq 0$ and a real $\alpha > 0$ such that

(i) $F'X + XF + TF'XF + H'H$
$+ (H'E + (I + TF)'XG)(\alpha^2 I - TG'XG - E'E)^{-1}(E'H + G'X(I + TF)) \leq -(2 - T\eta)\eta X$

(ii) $\alpha^2 I - TG'XG - E'E > 0,$

then

(a) the eigenvalues of F lie in D_T^η

(b) $\|T_{wz}\|_\infty \leq \alpha$.

Proof See Appendix B.1. □

Returning to Problem 3.2.2, we choose $u = K^c x$ and solve for the closed-loop system matrices in terms of the controller gain K^c. We find conditions guaranteeing conditions (i), (ii), and detectability in the bounded real lemma, resulting in the following theorem. The full derivation is given in Appendix B.2.

Theorem 3.3.1 *For the discrete-time plant*

$$\delta x = A_\delta x + B_\delta u + (\sqrt{N} G_\delta)\tilde{w}, \quad x(0) = 0,$$

with the performance-output variable

$$\tilde{z} = \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta x + D_{BH} u) + D_{GH}\tilde{w} \\ u \end{pmatrix},$$

with (A_δ, H_δ) detectable and with state feedback $u = K^c x$, a sufficient condition to guarantee that the closed-loop discrete-time system is stable and has H_∞ norm from $\tilde{w}$ to $\tilde{z}$ less than a given value α is

$$\begin{aligned}
K^c = -[&I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\
&+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
\cdot [&B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\
&+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)],
\end{aligned} \tag{3.14}$$

where

$$\Upsilon := \alpha^2 I - TNG_\delta' X G_\delta - D_{GH}' D_{GH},$$

and $X > 0$ satisfies

$$\begin{aligned}
0 = \; & A_\delta' X + X A_\delta + T A_\delta' X A_\delta + \tfrac{1}{N} H_\delta' H_\delta \\
&+ \left(\tfrac{1}{\sqrt{N}} H_\delta' D_{GH} + \sqrt{N} (I + TA_\delta)' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right) \\
&- [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]' \\
&\cdot [I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
&\cdot [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]
\end{aligned} \tag{3.15}$$

and

$$X^{-1} - TNG_\delta (\alpha^2 I - D_{GH}' D_{GH})^{-1} G_\delta' > 0.$$

If, in addition, the closed-loop system and observer are required to have degree of stability η, where the degree of stability of a system is defined as in Definition 2.5.1, we obtain the following controller design.

Theorem 3.3.2 *For the discrete-time plant*

$$\delta x = A_\delta x + B_\delta u + (\sqrt{N} G_\delta) \tilde{w}, \quad x(0) = 0,$$

with the performance-output variable

$$\tilde{z} = \begin{pmatrix} \frac{1}{\sqrt{N}} (H_\delta x + D_{BH} u) + D_{GH} \tilde{w} \\ u \end{pmatrix},$$

with all unobservable modes of (A_δ, H_δ) in D_T^η and with state feedback $u = K^c x$, a sufficient condition to guarantee that the closed-loop discrete-time system is stable with degree of stability η and has H_∞ norm from $\tilde{w}$ to $\tilde{z}$ less than a given value α is

$$\begin{aligned} K^c = -[&I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ &+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\ &\cdot [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\ &+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)], \end{aligned} \tag{3.16}$$

where

$$\Upsilon := \alpha^2 I - TNG_\delta' X G_\delta - D_{GH}' D_{GH},$$

and $X > 0$ satisfies

$$\begin{aligned} -(2 - T\eta)\eta X = {} & A_\delta' X + X A_\delta + T A_\delta' X A_\delta + \tfrac{1}{N} H_\delta' H_\delta \\ & + \left(\tfrac{1}{\sqrt{N}} H_\delta' D_{GH} + \sqrt{N}(I + TA_\delta)' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right) \\ & - [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\ & + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]' \\ & \cdot [I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ & + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\ & \cdot [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\ & + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)] \end{aligned} \tag{3.17}$$

and

$$X^{-1} - TNG_\delta(\alpha^2 I - D_{GH}' D_{GH})^{-1} G_\delta' > 0.$$

Proof This follows from Lemma 3.3.2. The derivation is the same as for Theorem 3.3.1. □

Now consider Problem 3.2.4. Given the form chosen for the controllers and observers, we can find the closed-loop system matrices in terms of the controller and observer gains. We then find conditions guaranteeing conditions (i), (ii), and detectability in the bounded real lemma, resulting in Theorem 3.3.3. The complete derivation is given in Appendix B.3.

The following notation is required to write the design equations in a manageable form.

$$u := \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix}, w := \begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix}, \quad K^c := \begin{pmatrix} K_1^c \\ \vdots \\ K_p^c \end{pmatrix}, \quad G_{\delta,C} := \begin{pmatrix} G_\delta \\ \vdots \\ G_\delta \end{pmatrix},$$

$$C_D := \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & C_p \end{pmatrix}, \quad K_D^c := \begin{pmatrix} K_1^c & 0 & \cdots & 0 \\ 0 & K_2^c & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K_p^c \end{pmatrix}, \quad K_D^o := \begin{pmatrix} K_1^o & 0 & \cdots & 0 \\ 0 & K_2^o & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K_p^o \end{pmatrix},$$

$$B_\delta := (B_\delta^1 \; B_\delta^2 \cdots B_\delta^p), \qquad D_{BH} := (D_{BH}^1 \; D_{BH}^2 \cdots D_{BH}^p),$$

and

$$A_e := \begin{pmatrix} A_\delta + B_\delta K^c + \sqrt{N} G_\delta K^d & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_\delta + B_\delta K^c + \sqrt{N} G_\delta K^d \end{pmatrix}$$

$$+ \begin{pmatrix} -B_\delta^1 K_1^c & -B_\delta^2 K_2^c & \cdots & -B_\delta^p K_p^c \\ -B_\delta^1 K_1^c & \ddots & & \vdots \\ \vdots & & \ddots & -B_\delta^p K_p^c \\ -B_\delta^1 K_1^c & \cdots & -B_\delta^{p-1} K_{p-1}^c & -B_\delta^p K_p^c \end{pmatrix}.$$

Theorem 3.3.3 *For the discrete-time decentralized plant*

$$\delta x = A_\delta x + \sum_{i=1}^{p} B_\delta^i u_i + (\sqrt{N} G_\delta) \tilde{w}, \quad x(0) = 0,$$

with the performance-output variable

$$\tilde{z} = \begin{pmatrix} \frac{1}{\sqrt{N}} \left(H_\delta x + \sum_{j=1}^{p} D_{BH}^j u_j \right) + D_{GH} \tilde{w} \\ u \end{pmatrix},$$

with (A_δ, H_δ) detectable and with observer-based decentralized discrete-time controller $u_i = K_i^c \xi_i$, where

$$\delta \xi_i = \left(A_\delta + B_\delta^i K_i^c + \sum_{j \neq i} B_\delta^j K_j^c + (\sqrt{N} G_\delta) K^d \right) \xi_i + K_i^o (y_i - C_i \xi_i), \quad \xi_i(0) = 0,$$

a sufficient condition to guarantee that the closed-loop system is stable and has H_∞ norm from $\binom{\tilde{w}}{w}$ to $\tilde{z}$ less than a given value α is

$$\begin{aligned} K^c = -[&I + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ &+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta \right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta \right)]^{-1} \\ \cdot [&B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\ &+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta \right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} T G_\delta' X (I + T A_\delta) \right)], \end{aligned}$$

$$K^d = \Upsilon^{-1}\left(\frac{1}{\sqrt{N}}D'_{GH}(H_\delta + D_{BH}K^c) + \sqrt{N}G'_\delta X(I + T(A_\delta + B_\delta K^c))\right), \tag{3.18}$$

where

$$\Upsilon := \alpha^2 I - TNG'_\delta X G_\delta - D'_{GH}D_{GH},$$

and K^o_D block diagonal, $X > 0$, and $W > 0$ satisfy

$$\begin{aligned}
0 = {} & A'_\delta X + XA_\delta + TA'_\delta X A_\delta + \tfrac{1}{N}H'_\delta H_\delta \\
& + \left(\tfrac{1}{\sqrt{N}}H'_\delta D_{GH} + \sqrt{N}(I + TA_\delta)'XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}H_\delta + \sqrt{N}G'_\delta X(I + TA_\delta)\right) \\
& - [B'_\delta X(I + TA_\delta) + \tfrac{1}{N}D'_{BH}H_\delta \\
& \quad + \left(\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}H_\delta + \sqrt{N}G'_\delta X(I + TA_\delta)\right)]' \\
& \cdot [I + TB'_\delta XB_\delta + \tfrac{1}{N}D'_{BH}D_{BH} \\
& \quad + \left(\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_\delta XB_\delta\right)]^{-1} \\
& \cdot [B'_\delta X(I + TA_\delta) + \tfrac{1}{N}D'_{BH}H_\delta \\
& \quad + \left(\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}H_\delta + \sqrt{N}G'_\delta X(I + TA_\delta)\right)]
\end{aligned}$$

and

$$\begin{aligned}
W = {} & TNG_{\delta,C}\Upsilon^{-1}G'_{\delta,C} + T\tfrac{1}{\alpha^2}K^o_D K^{o\prime}_D \\
& + [I + T(A_e - G_{\delta,C}\Upsilon^{-1}D'_{GH}D_{BH}K^c_D - TNG_{\delta,C}\Upsilon^{-1}G'_\delta XB_\delta K^c_D) - TK^o_D C_D] \\
& \cdot (W^{-1} - TK^{o\prime}_D[I + TB'_\delta XB_\delta + \tfrac{1}{N}D'_{BH}D_{BH} \\
& \quad + \left(\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_\delta XB_\delta\right)]K^c_D)^{-1} \\
& \cdot [I + T(A_e - G_{\delta,C}\Upsilon^{-1}D'_{GH}D_{BH}K^c_D - TNG_{\delta,C}\Upsilon^{-1}G'_\delta XB_\delta K^c_D) - TK^o_D C_D]',
\end{aligned} \tag{3.19}$$

such that the eigenvalues of $A_e - K^o_D C_D$ are in the stability region and

$$X^{-1} - TNG_\delta(\alpha^2 I - D'_{GH}D_{GH})^{-1}G'_\delta > 0,$$

$$\begin{aligned}
& W^{-1} - TK^{o\prime}_D[I + TB'_\delta XB_\delta + \tfrac{1}{N}D'_{BH}D_{BH} \\
& \quad + (\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta)\Upsilon^{-1}(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_\delta XB_\delta)]K^c_D > 0,
\end{aligned}$$

and

$$|I + T(A_e - G_{\delta,C}\Upsilon^{-1}D'_{GH}D_{BH}K^c_D - TNG_{\delta,C}\Upsilon^{-1}G'_\delta XB_\delta K^c_D) - TK^o_D C_D| \neq 0.$$

Note that the state-feedback gain K^c and the design equation for X have the same form as in the state-feedback case.

The choice of $K^o_D = \alpha^2 W_D C'_D$, where W_D is the block diagonal part of W, results in one solution.

The theorem can be modified as follows to guarantee a given degree of stability for the resulting closed-loop system:

Theorem 3.3.4 *For the discrete-time decentralized plant*

$$\delta x = A_\delta x + \sum_{i=1}^{p} B_\delta^i u_i + (\sqrt{N} G_\delta)\tilde{w}, \quad x(0) = 0,$$

with the performance-output variable

$$\tilde{z} = \begin{pmatrix} \frac{1}{\sqrt{N}} \left(H_\delta x + \sum_{j=1}^{p} D_{BH}^j u_j \right) + D_{GH}\tilde{w} \\ u \end{pmatrix},$$

with all unobservable modes of (A_δ, H_δ) in D_T^η and with observer-based decentralized discrete-time controller $u_i = K_i^c \xi_i$, where

$$\delta\xi_i = \left(A_\delta + B_\delta^i K_i^c + \sum_{j \neq i} B_\delta^j K_j^c + (\sqrt{N} G_\delta) K^d \right) \xi_i + K_i^o (y_i - C_i \xi_i), \quad \xi_i(0) = 0,$$

a sufficient condition to guarantee that the closed-loop system has degree of stability η and H_∞ norm from $\binom{\tilde{w}}{w}$ to $\tilde{z}$ less than a given value α is

$$\begin{aligned}
K^c = &-[I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
&\cdot [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} T G_\delta' X (I + TA_\delta)\right)],
\end{aligned}$$

$$K^d = \Upsilon^{-1} \left(\frac{1}{\sqrt{N}} D_{GH}' (H_\delta + D_{BH} K^c) + \sqrt{N} G_\delta' X (I + T(A_\delta + B_\delta K^c)) \right), \tag{3.20}$$

where

$$\Upsilon := \alpha^2 I - TNG_\delta' X G_\delta - D_{GH}' D_{GH},$$

and K_D^o block diagonal, $X > 0$, and $W > 0$ satisfy

$$\begin{aligned}
-(2 - T\eta)\eta X = &\; A_\delta' X + X A_\delta + T A_\delta' X A_\delta + \tfrac{1}{N} H_\delta' H_\delta \\
&\quad + \left(\tfrac{1}{\sqrt{N}} H_\delta' D_{GH} + \sqrt{N} (I + TA_\delta)' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right) \\
&- [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]' \\
&\cdot [I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
&\cdot [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]
\end{aligned}$$

and

$$
\begin{aligned}
W = {} & TNG_{\delta,C}\Upsilon^{-1}G'_{\delta,C} + T\tfrac{1}{\alpha^2}K^{\circ}_{D}K^{\circ\prime}_{D} \\
& + [I + T(A_e - G_{\delta,C}\Upsilon^{-1}D'_{GH}D_{BH}K^{c}_{D} - TNG_{\delta,C}\Upsilon^{-1}G'_{\delta}XB_{\delta}K^{c}_{D}) - TK^{\circ}_{D}C_{D}] \\
& \quad \cdot((1-T\eta)^2W^{-1} - TK^{c\prime}_{D}[I + TB'_{\delta}XB_{\delta} + \tfrac{1}{N}D'_{BH}D_{BH} \\
& \quad\quad + \left(\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_{\delta}XG_{\delta}\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_{\delta}XB_{\delta}\right)]K^{c}_{D})^{-1} \\
& \quad \cdot[I + T(A_e - G_{\delta,C}\Upsilon^{-1}D'_{GH}D_{BH}K^{c}_{D} - TNG_{\delta,C}\Upsilon^{-1}G'_{\delta}XB_{\delta}K^{c}_{D}) - TK^{\circ}_{D}C_{D}]',
\end{aligned}
\tag{3.21}
$$

such that the eigenvalues of $A_e - K^{\circ}_{D}C_D$ are in D^{η}_{T} and

$$X^{-1} - TNG_{\delta}(\alpha^2I - D'_{GH}D_{GH})^{-1}G'_{\delta} > 0,$$

$$
\begin{aligned}
(1-T\eta)^2W^{-1} & - TK^{c\prime}_{D}[I+TB'_{\delta}XB_{\delta}+\tfrac{1}{N}D'_{BH}D_{BH} \\
& + (\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_{\delta}XG_{\delta})\Upsilon^{-1}(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_{\delta}XB_{\delta})]K^{c}_{D} > 0,
\end{aligned}
$$

and

$$|I + T(A_e - G_{\delta,C}\Upsilon^{-1}D'_{GH}D_{BH}K^{c}_{D} - TNG_{\delta,C}\Upsilon^{-1}G'_{\delta}XB_{\delta}K^{c}_{D}) - TK^{\circ}_{D}C_{D}| \neq 0.$$

Proof This follows from Lemma 3.3.2. The derivation is the same as for Theorem 3.3.3. □

3.4 Design Equations for Sampled-data Systems

Some of the matrices that were defined for the two-rate problems increase in dimension as N increases. However, the elements of interest are the values of the products of matrices in the design equations, not the individual matrices themselves. To reduce these designs to the sampled-data designs for Problems 3.2.1 and 3.2.3, we first evaluate the limit of the matrix products in the design equations.

Note first that A_δ and B_δ are not functions of N. Now consider the other terms:

$$\tfrac{1}{N}H'_{\delta}H_{\delta} = \tfrac{1}{T}\left(\tfrac{T}{N}\textstyle\sum_{i=0}^{N-1} e^{A'_c(i\frac{T}{N})}H'_cH_ce^{A_c(i\frac{T}{N})}\right) \longrightarrow \tfrac{1}{T}\int_0^T e^{A'_ct}H'_cH_ce^{A_ct}\,dt =: M_1 \quad \text{as } N \to \infty$$

since the expression in parentheses is a Riemann sum.

$$
\begin{aligned}
\tfrac{1}{N}D'_{BH}H_{\delta} &= \tfrac{1}{T}\left(\tfrac{T}{N}\textstyle\sum_{i=1}^{N-1} B'_c\left(\int_0^{i\frac{T}{N}} e^{A'_cs}\,ds\right)H'_cH_ce^{A_c(i\frac{T}{N})}\right) \\
&\longrightarrow \tfrac{1}{T}\int_0^T B'_c\left(\int_0^t e^{A'_cs}\,ds\right)H'_cH_ce^{A_ct}\,dt =: M_2.
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{1}{N}D'_{BH}D_{BH} &= \tfrac{1}{T}\left(\tfrac{T}{N}\textstyle\sum_{i=1}^{N-1} B'_c\left(\int_0^{i\frac{T}{N}} e^{A'_cs}\,ds\right)H'_cH_c\left(\int_0^{i\frac{T}{N}} e^{A_c\sigma}\,d\sigma\right)B_c\right) \\
&\longrightarrow \tfrac{1}{T}\int_0^T B'_c\left(\int_0^t e^{A'_cs}\,ds\right)H'_cH_c\left(\int_0^t e^{A_c\sigma}\,d\sigma\right)B_c\,dt =: M_3.
\end{aligned}
$$

To find the limits of the remaining terms, we first examine

$$\Upsilon(N)^{-1} = (\alpha^2 I_N - NTG_\delta' X G_\delta - D_{GH}' D_{GH})^{-1},$$

an $N \times N$ dimensional matrix that appears in the middle of the remaining products. First, note that

$$\Upsilon(N) = \alpha^2 I_N - NTG_\delta' X G_\delta - D_{GH}' D_{GH}$$

$$= \alpha^2 I_N$$

$$-T\frac{1}{N}\begin{pmatrix} G_c' e^{A_c'(\frac{N-1}{N})T}\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c' t}\,dt\right) \\ \vdots \\ G_c'\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c' t}\,dt\right)\end{pmatrix} X \left(e^{A_c(\frac{N-1}{N})T}\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c t}\,dt\right)G_c \ldots \left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c t}\,dt\right)G_c\right)$$

$$-T^2\frac{1}{N^2}\begin{pmatrix} 0 & & \cdots & 0 \\ H_c\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c t}\,dt\right)G_c & & & \vdots \\ \vdots & & \ddots & \\ H_c e^{A_c(\frac{N-2}{N})T}\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c t}\,dt\right)G_c & \ldots & H_c\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c t}\,dt\right)G_c & 0\end{pmatrix}'$$

$$\cdot\begin{pmatrix} 0 & & \cdots & 0 \\ H_c\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c t}\,dt\right)G_c & & & \vdots \\ \vdots & & \ddots & \\ H_c e^{A_c(\frac{N-2}{N})T}\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c t}\,dt\right)G_c & \ldots & H_c\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c t}\,dt\right)G_c & 0\end{pmatrix}.$$

The second and third matrix terms are $o(\frac{1}{N})$ and $o(\frac{1}{N^2})$, respectively, uniformly in the elements of the matrices, as long as X is $O(1)$. Thus, the inverse $\Upsilon(N)^{-1}$ will equal $\frac{1}{\alpha^2}I_N$ plus terms that are $O(\frac{1}{N})$ and $o(\frac{1}{N^2})$. The $o(\frac{1}{N^2})$ terms do not affect the resulting products: the products will tend to zero as $N \to \infty$. However, the $O(\frac{1}{N})$ terms do have an impact on the products.

We invert $\Upsilon(N)$ to isolate the $O(\frac{1}{N})$ terms as follows:

$$\begin{aligned}\Upsilon(N)^{-1} &= (\alpha^2 I_N - D_{GH}' D_{GH} - NTG_\delta' X G_\delta)^{-1} \\ &= (\alpha^2 I_N - D_{GH}' D_{GH})^{-1} + (\alpha^2 I_N - D_{GH}' D_{GH})^{-1} \\ &\quad \cdot NTG_\delta' \left(X^{-1} - NTG_\delta(\alpha^2 I_N - D_{GH}' D_{GH})^{-1} G_\delta'\right)^{-1} G_\delta(\alpha^2 I_N - D_{GH}' D_{GH})^{-1}.\end{aligned}$$

The inverse of $\alpha^2 I_N - D_{GH}' D_{GH}$ is $\frac{1}{\alpha^2}I_N$ plus $o(\frac{1}{N^2})$ terms, so it can be treated in products as $\frac{1}{\alpha^2}I_N$.

The following products with the $O(1)$ and $O(\frac{1}{N})$ terms in the expansion converge to finite limits:

$$\frac{1}{N}H'_\delta D_{GH}D'_{GH}H_\delta = \frac{1}{T}\,\frac{T}{N}\sum_{j=1}^{N-1}\left(\frac{T}{N}\sum_{i=j}^{N-1} e^{A'_c(\frac{i}{N})T}H'_cH_ce^{A_c(\frac{i-j}{N})T}\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c\sigma}\,d\sigma\right)G_c\right)$$
$$\cdot\left(\frac{T}{N}\sum_{l=j}^{N-1} G'_c\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A'_c\phi}\,d\phi\right)e^{A'_c(\frac{l-j}{N})T}H'_cH_ce^{A_c(\frac{l}{N})T}\right)$$
$$\longrightarrow \frac{1}{T}\int_0^T\left(\int_t^T e^{A'_cs}H'_cH_ce^{A_c(s-t)}G_c\,ds\right)\left(\int_t^T G'_ce^{A'_c(\sigma-t)}H'_cH_ce^{A_c\sigma}\,d\sigma\right)\,dt =: M_4.$$

$$H'_\delta D_{GH}G'_\delta = \frac{1}{T}\,\frac{T}{N}\sum_{j=1}^{N-1}\frac{T}{N}\sum_{i=j}^{N-1} e^{A'_c(\frac{i}{N})T}H'_cH_ce^{A_c(\frac{i-j}{N})T}\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c\sigma}\,d\sigma\right)G_c$$
$$\cdot G'_c\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A'_c\phi}\,d\phi\right)e^{A'_c(\frac{N-j}{N})T}$$
$$\longrightarrow \frac{1}{T}\int_0^T\int_t^T e^{A'_cs}H'_cH_ce^{A_c(s-t)}G_cG'_ce^{A'_c(T-t)}\,ds\,dt =: M_5.$$

$$NG_\delta G'_\delta = \frac{1}{T}\left(\frac{T}{N}\sum_{i=1}^{N} e^{A_c(\frac{N-i}{N})T}\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c\sigma}\,d\sigma\right)G_cG'_c\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A'_c\phi}\,d\phi\right)e^{A'_c(\frac{N-i}{N})T}\right)$$
$$\longrightarrow \frac{1}{T}\int_0^T e^{A_c(T-t)}G_cG'_ce^{A'_c(T-t)}\,dt =: M_6.$$

$$\frac{1}{N}D'_{BH}D_{GH}D'_{GH}D_{BH}$$
$$= \frac{1}{T}\,\frac{T}{N}\sum_{j=1}^{N-1}\left(\frac{T}{N}\sum_{i=j}^{N-1} B'_c\left(\int_0^{i\frac{T}{N}} e^{A'_c\sigma}\,d\sigma\right)H'_cH_ce^{A_c(\frac{i-j}{N})T}\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c\rho}\,d\rho\right)G_c\right)$$
$$\cdot\left(\frac{T}{N}\sum_{l=j}^{N-1} G'_c\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A'_c\phi}\,d\phi\right)e^{A'_c(\frac{l-j}{N})T}H'_cH_c\left(\int_0^{l\frac{T}{N}} e^{A_c\psi}\,d\psi\right)B_c\right)$$
$$\longrightarrow \frac{1}{T}\int_0^T\left(B'_c\int_t^T\left(\int_0^s e^{A'_c\phi}\,d\phi\right)H'_cH_ce^{A_c(s-t)}G_c\,ds\right)\left(\int_t^T G'_ce^{A'_c(\sigma-t)}H'_cH_c\left(\int_0^\sigma e^{A_c\psi}\,d\psi\right)d\sigma\,B_c\right)dt$$
$$=: M_7.$$

$$\frac{1}{N}D'_{BH}D_{GH}D'_{GH}H_\delta$$
$$= \frac{1}{T}\,\frac{T}{N}\sum_{j=1}^{N-1}\left(\frac{T}{N}\sum_{i=j}^{N-1} B'_c\left(\int_0^{i\frac{T}{N}} e^{A'_c\phi}\,d\phi\right)H'_cH_ce^{A_c(\frac{i-j}{N})T}\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c\rho}\,d\rho\right)G_c\right)$$
$$\cdot\left(\frac{T}{N}\sum_{l=j}^{N-1} G'_c\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A'_c\psi}\,d\psi\right)e^{A'_c(\frac{l-j}{N})T}H'_cH_ce^{A_c(\frac{l}{N})T}\right)$$
$$\longrightarrow \frac{1}{T}\int_0^T\left(\int_t^T B'_c\left(\int_0^s e^{A'_c\phi}\,d\phi\right)H'_cH_ce^{A_c(s-t)}G_c\,ds\right)\left(\int_t^T G'_ce^{A'_c(\sigma-t)}H'_cH_ce^{A_c\sigma}\,d\sigma\right)\,dt =: M_8.$$

$$D'_{BH}D_{GH}G'_\delta = \frac{1}{T}\frac{T}{N}\sum_{j=1}^{N-1}\frac{T}{N}\sum_{i=j}^{N-1} B'_c\left(\int_0^{i\frac{T}{N}} e^{A'_c\sigma}\,d\sigma\right)H'_cH_ce^{A_c(\frac{i-j}{N})T}\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A_c\phi}\,d\phi\right)G_c$$
$$\cdot G'_c\left(\frac{N}{T}\int_0^{\frac{T}{N}} e^{A'_c\psi}\,d\psi\right)e^{A'_c(\frac{N-j}{N})T}$$
$$\longrightarrow \frac{1}{T}\int_0^T\int_t^T B'_c\left(\int_0^s e^{A'_c\phi}\,d\phi\right)H'_cH_ce^{A_c(s-t)}G_cG'_ce^{A'_c(T-t)}\,ds\,dt =: M_9.$$

$$TH'_\delta D_{GH}G'_\delta(X^{-1} - NTG_\delta(\alpha^2I_N - D'_{GH}D_{GH})^{-1}G'_\delta)^{-1}G_\delta D'_{GH}H_\delta$$
$$\longrightarrow \frac{1}{T}\int_0^T\int_t^T e^{A'_cs}H'_cH_ce^{-A_c(t-s)}G_cG'_ce^{A'_c(T-t)}\,ds\,dt$$
$$\cdot\left(X^{-1} - \frac{1}{\alpha^2}\int_0^T e^{A_c(T-t)}G_cG'_ce^{A'_c(T-t)}\,dt\right)^{-1}$$
$$\cdot\int_0^T\int_t^T e^{A_c(T-t)}G_cG'_ce^{-A'_c(t-s)}H'_cH_ce^{A_cs}\,ds\,dt$$
$$= TM_5(X^{-1} - \tfrac{1}{\alpha^2}TM_6)^{-1}M'_5,$$

$$TNH_\delta' D_{GH} G_\delta'(X^{-1} - NTG_\delta(\alpha^2 I_N - D_{GH}' D_{GH})^{-1} G_\delta')^{-1} G_\delta G_\delta'$$
$$\longrightarrow \tfrac{1}{T}\int_0^T \int_t^T e^{A_c' s} H_c' H_c e^{-A_c(t-s)} G_c G_c' e^{A_c'(T-t)}\, ds\, dt$$
$$\cdot \left(X^{-1} - \tfrac{1}{\alpha^2}\int_0^T e^{A_c(T-t)} G_c G_c' e^{A_c'(T-t)}\, dt\right)^{-1}$$
$$\cdot \int_0^T e^{A_c(T-t)} G_c G_c' e^{A_c'(T-t)}\, dt$$
$$= TM_5(X^{-1} - \tfrac{1}{\alpha^2} TM_6)^{-1} M_6,$$

$$TN^2 G_\delta G_\delta'(X^{-1} - NTG_\delta(\alpha^2 I_N - D_{GH}' D_{GH})^{-1} G_\delta')^{-1} G_\delta G_\delta'$$
$$\longrightarrow \tfrac{1}{T}\int_0^T e^{A_c(T-t)} G_c G_c' e^{A_c'(T-t)}\, dt$$
$$\cdot \left(X^{-1} - \tfrac{1}{\alpha^2}\int_0^T e^{A_c(T-t)} G_c G_c' e^{A_c'(T-t)}\, dt\right)^{-1}$$
$$\cdot \int_0^T e^{A_c(T-t)} G_c G_c' e^{A_c'(T-t)}\, dt$$
$$= TM_6(X^{-1} - \tfrac{1}{\alpha^2} TM_6)^{-1} M_6,$$

$$TD_{BH}' D_{GH} G_\delta'(X^{-1} - NTG_\delta(\alpha^2 I_N - D_{GH}' D_{GH})^{-1} G_\delta')^{-1} G_\delta D_{GH}' H_\delta$$
$$\longrightarrow \tfrac{1}{T}\int_0^T \int_t^T B_c' \left(\int_0^s e^{A_c'\sigma}\, d\sigma\right) H_c' H_c e^{-A_c(t-s)} G_c G_c' e^{A_c'(T-t)}\, ds\, dt$$
$$\cdot \left(X^{-1} - \tfrac{1}{\alpha^2}\int_0^T e^{A_c(T-t)} G_c G_c' e^{A_c'(T-t)}\, dt\right)^{-1}$$
$$\cdot \int_0^T \int_t^T e^{A_c(T-t)} G_c G_c' e^{-A_c'(t-s)} H_c' H_c e^{A_c s}\, ds\, dt$$
$$= TM_9(X^{-1} - \tfrac{1}{\alpha^2} TM_6)^{-1} M_5',$$

$$TND_{BH}' D_{GH} G_\delta'(X^{-1} - NTG_\delta(\alpha^2 I_N - D_{GH}' D_{GH})^{-1} G_\delta')^{-1} G_\delta G_\delta'$$
$$\longrightarrow \tfrac{1}{T}\int_0^T \int_t^T B_c' \left(\int_0^s e^{A_c'\sigma}\, d\sigma\right) H_c' H_c e^{-A_c(t-s)} G_c G_c' e^{A_c'(T-t)}\, ds\, dt$$
$$\cdot \left(X^{-1} - \tfrac{1}{\alpha^2}\int_0^T e^{A_c(T-t)} G_c G_c' e^{A_c'(T-t)}\, dt\right)^{-1}$$
$$\cdot \int_0^T e^{A_c(T-t)} G_c G_c' e^{A_c'(T-t)}\, dt$$
$$= TM_9(X^{-1} - \tfrac{1}{\alpha^2} TM_6)^{-1} M_6,$$

$$TND_{BH}' D_{GH} G_\delta'(X^{-1} - NTG_\delta(\alpha^2 I_N - D_{GH}' D_{GH})^{-1} G_\delta')^{-1} G_\delta D_{GH}' D_{BH}$$
$$\longrightarrow \tfrac{1}{T}\int_0^T \int_t^T B_c' \left(\int_0^s e^{A_c'\sigma}\, d\sigma\right) H_c' H_c e^{-A_c(t-s)} G_c G_c' e^{A_c'(T-t)}\, ds\, dt$$
$$\cdot \left(X^{-1} - \tfrac{1}{\alpha^2}\int_0^T e^{A_c(T-t)} G_c G_c' e^{A_c'(T-t)}\, dt\right)^{-1}$$
$$\cdot \int_0^T \int_t^T e^{A_c(T-t)} G_c G_c' e^{-A_c'(t-s)} H_c' H_c \left(\int_0^s e^{A_c\sigma}\, d\sigma\right) B_c\, ds\, dt$$
$$= TM_9(X^{-1} - \tfrac{1}{\alpha^2} TM_6)^{-1} M_9'.$$

Thus, we can take the limits of the expressions as $N \to \infty$ for the controller in Theorem 3.3.1. Also, note that, except for M_1 and M_6, all of the other terms M_1–M_9 tend to zero as $T \to 0$. The exceptions $M_1 \to H_c' H_c$ and $M_6 \to G_c G_c'$ as $T \to 0$. Thus, the expressions for the controller gain and design equation are close to those of the continuous-time optimal gain and Riccati equation, agreeing with them in the limit as $T \to 0$. For small enough T, the solutions to these equations exist if the continuous-time solutions exist.

Next, we state and prove the following theorem for the solution to Problem 3.2.1.

Theorem 3.4.1 *Given the continuous-time plant*

$$\dot{x} = A_c x + B_c u + G_c w_0, \quad x(0) = 0,$$

and performance variable

$$z = \begin{pmatrix} H_c x \\ u \end{pmatrix},$$

with (A_c, H_c) detectable, then the controller

$$u(t) = K^c x(kT), \quad t \in [kT, (k+1)T); \qquad k = 0, 1, 2, \ldots,$$

where

$$\begin{aligned} K^c = -\{&I + TB_\delta' X B_\delta + M_3 + \tfrac{1}{\alpha^2}[M_7 + TM_9 X B_\delta + TB_\delta' X M_9' + T^2 B_\delta' X M_6 X B_\delta] \\ &+ \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB_\delta' X M_6)\left(X^{-1} - \tfrac{1}{\alpha^2} T M_6\right)^{-1} (M_9' + TM_6 X B_\delta)\}^{-1} \\ &\cdot \{B_\delta' X(I + TA_\delta) + M_2 + \tfrac{1}{\alpha^2}[M_8 + M_9 X(I + TA_\delta) + TB_\delta' X M_5' + TB_\delta' X M_6 X(I + TA_\delta)] \\ &+ \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB_\delta' X M_6)\left(X^{-1} - \tfrac{1}{\alpha^2} T M_6\right)^{-1} (M_5' + M_6 X(I + TA_\delta))\} \end{aligned} \tag{3.22}$$

and $X > 0$ satisfies

$$\begin{aligned} -(2 - T\eta)\eta X = {} & A_\delta' X + X A_\delta + T A_\delta' X A_\delta + M_1 \\ &+ \tfrac{1}{\alpha^2}[M_4 + M_5 X(I + TA_\delta) + (I + TA_\delta)' X M_5' + (I + TA_\delta)' X M_6 X (I + TA_\delta)] \\ &+ \left(\tfrac{1}{\alpha^2}\right)^2 T(M_5 + (I + TA_\delta)' X M_6)\left(X^{-1} - \tfrac{1}{\alpha^2} T M_6\right)^{-1} (M_5' + M_6 X(I + TA_\delta)) \\ &- \{B_\delta' X(I + TA_\delta) + M_2 + \tfrac{1}{\alpha^2}[M_8 + M_9 X(I + TA_\delta) + TB_\delta' X M_5' + TB_\delta' X M_6 X(I + TA_\delta)] \\ &+ \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB_\delta' X M_6)\left(X^{-1} - \tfrac{1}{\alpha^2} T M_6\right)^{-1} (M_5' + M_6 X(I + TA_\delta))\}' \\ &\cdot \{I + TB_\delta' X B_\delta + M_3 + \tfrac{1}{\alpha^2}[M_7 + TM_9 X B_\delta + TB_\delta' X M_9' + T^2 B_\delta' X M_6 X B_\delta] \\ &+ \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB_\delta' X M_6)\left(X^{-1} - \tfrac{1}{\alpha^2} T M_6\right)^{-1} (M_9' + TM_6 X B_\delta)\}^{-1} \\ &\cdot \{B_\delta' X(I + TA_\delta) + M_2 + \tfrac{1}{\alpha^2}[M_8 + M_9 X(I + TA_\delta) + TB_\delta' X M_5' + TB_\delta' X M_6 X(I + TA_\delta)] \\ &+ \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB_\delta' X M_6)\left(X^{-1} - \tfrac{1}{\alpha^2} T M_6\right)^{-1} (M_5' + M_6 X(I + TA_\delta))\} \end{aligned} \tag{3.23}$$

where $X^{-1} - \frac{1}{\alpha^2} T M_6 = X^{-1} - \frac{1}{\alpha^2} \int_0^T e^{A_c(T-t)} G_c G_c' e^{A_c'(T-t)}\, dt > 0$, and

$$\begin{aligned} &I + TB_\delta' X B_\delta + M_3 + \tfrac{1}{\alpha^2}[M_7 + TM_9 X B_\delta + TB_\delta' X M_9' + T^2 B_\delta' X M_6 X B_\delta] \\ &\quad + \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB_\delta' X M_6)\left(X^{-1} - \tfrac{1}{\alpha^2} T M_6\right)^{-1} (M_9' + TM_6 X B_\delta) > 0, \end{aligned}$$

where η is less than the degree of stability of the unobservable modes of (A_c, H_c), results in a stable closed-loop continuous-time system with Riemann H_∞ norm from w_0 to z less than α.

Proof

Let $w(t)$ be any Riemann-integrable signal in $\mathcal{L}_2$ not identically zero. Then there exists some finite real number $c > 0$ such that

$$\int_0^\infty w(t)'w(t)\,dt < c.$$

Since $w(t)'w(t)$ is Riemann integrable, the Riemann sums

$$\frac{T}{N}\sum_{i=0}^{\infty} w(i\frac{T}{N})'w(i\frac{T}{N}) \longrightarrow \int_0^\infty w(t)'w(t)\,dt$$

as $N \to \infty$. Let $C > c$. Then there exists $N_1 > 0$, such that, for all $N > N_1$,

$$\frac{T}{N}\sum_{i=0}^{\infty} w(i\frac{T}{N})'w(i\frac{T}{N}) < C.$$

Thus, for all $N > N_1$,

$$\left\{w(i\frac{T}{N})\right\}_{i=0,1,\ldots} \in \ell_2.$$

By Theorem 3.3.1, for each $N > N_1$,

$$\frac{\frac{T}{N}\sum_{i=0}^{\infty} z_{K^c_N}(i\frac{T}{N})'z_{K^c_N}(i\frac{T}{N})}{\frac{T}{N}\sum_{i=0}^{\infty} w(i\frac{T}{N})'w(i\frac{T}{N})} < \alpha$$

where $z_{K^c_N}(t)$ is the regulated output of the closed-loop system (3.1), (3.2), (3.3) with the control gain $K^c = K^c_N$.

Note that the equation for K^c, (3.22), and the design equation, (3.23), written as $0 = F(X, M)$ where M is a variable containing all of the coefficient matrices M_1–M_9, are C^∞ functions of X and M if

$$\begin{aligned} I + TB_\delta'XB_\delta + M_3 + \tfrac{1}{\alpha^2}[M_7 + TM_9XB_\delta + TB_\delta'XM_9' + T^2B_\delta'XM_6XB_\delta] \\ + \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB_\delta'XM_6)\left(X^{-1} - \tfrac{1}{\alpha^2}TM_6\right)^{-1}(M_9' + TM_6XB_\delta) > 0. \end{aligned}$$

Applying the implicit function theorem by verifying that $\left|\frac{\partial F}{\partial X}(X_\infty, M)\right| \neq 0$, it can be shown that, if the design Equation (3.23) has a solution $X_\infty > 0$, then, in a neighborhood of $X_\infty > 0$ and M_1–M_9, the solution X is a continuous function of the coefficient matrices.

Since the expression for K^c (3.22) is continuous in both its coefficients and in X, for any $\epsilon > 0$, there exist $\delta > 0$ and $N_2(\epsilon) > 0$, such that, for $N > N_2$ and $\|X - X_\infty\| < \delta$, $\|K^c_N - K^c_\infty\| < \epsilon$. Since the coefficient matrices are the limits of Riemann sums as $N \to \infty$, for any $\delta > 0$, there exists

$N_3(\delta) > 0$, such that, for all $N > N_3$, $\|X_N - X_\infty\| < \delta$. Thus, given $\epsilon > 0$, $\|K_N^c - K_\infty^c\| < \epsilon$ for every $N > \max(N_2, N_3)$.

Since, for all N, $A_\delta + B_\delta K_N^c$ is stable with degree of stability η and since $K_N^c \to K_\infty^c$, there exists $N_4 > 0$, such that, for $N > N_4$, $A_\delta + B_\delta K_\infty^c$ is stable. Let $\varepsilon > 0$. Since $A_\delta + B_\delta K_N^c$ is stable with a given degree of stability η for each N, there exists P_N such that

$$\left|\frac{T}{N}\sum_{i=P_N}^{\infty} z_{K_N^c}(i\frac{T}{N})' z_{K_N^c}(i\frac{T}{N})\right| < \varepsilon/3.$$

Let $P = \sup_N P_N$. (Because of all the controllers have the same degree of stability, the supremum is finite.) Then,

$$\left|\frac{T}{N}\sum_{i=P}^{\infty} z_{K_N^c}(i\frac{T}{N})' z_{K_N^c}(i\frac{T}{N})\right| < \varepsilon/3$$

and, since $K^c \to K_\infty^c$ and K_∞^c also has degree of stability η,

$$\left|\frac{T}{N}\sum_{i=P}^{\infty} z_{K_\infty^c}(i\frac{T}{N})' z_{K_\infty^c}(i\frac{T}{N})\right| < \varepsilon/3.$$

Since the discrete trajectories $z_{K^c}(i\frac{T}{N})$ vary continuously with K^c, and since $K^c \to K_\infty^c$, there exists $N_5(\varepsilon) > N_4 > 0$, such that, for all $N > \max(N_5, N_1)$,

$$\left|\frac{T}{N}\sum_{i=0}^{P-1} z_{K_N^c}(i\frac{T}{N})' z_{K_N^c}(i\frac{T}{N}) - \frac{T}{N}\sum_{i=0}^{P-1} z_{K_\infty^c}(i\frac{T}{N})' z_{K_\infty^c}(i\frac{T}{N})\right| < \varepsilon/3.$$

Thus, for $N > \max(N_5, N_1)$,

$$\frac{\frac{T}{N}\sum_{i=0}^{\infty} z_{K_\infty^c}(i\frac{T}{N})' z_{K_\infty^c}(i\frac{T}{N})}{\frac{T}{N}\sum_{i=0}^{\infty} w(i\frac{T}{N})' w(i\frac{T}{N})} < \alpha + \frac{\varepsilon}{\frac{T}{N}\sum_{i=0}^{\infty} w(i\frac{T}{N})' w(i\frac{T}{N})}.$$

However,

$$\frac{T}{N}\sum_{i=0}^{\infty} w(i\frac{T}{N})' w(i\frac{T}{N})$$

converges to a constant s, and thus, for large N, this is bounded by $\alpha + \varepsilon'$, where ε' is arbitrary.

Since $z_{K_\infty^c}(t)$ is continuous in t, the Riemann sums above converge to integrals as $N \to \infty$. Thus,

$$\frac{\int_{i=0}^{\infty} z_{K_\infty^c}(t)' z_{K_\infty^c}(t)}{\int_{i=0}^{\infty} w(t)' w(t)} \le \alpha + \varepsilon'.$$

However, this holds for every $\varepsilon' > 0$. Thus,

$$\frac{\int_{i=0}^{\infty} z_{K_\infty^c}(t)' z_{K_\infty^c}(t)}{\int_{i=0}^{\infty} w(t)' w(t)} \le \alpha.$$

The disturbance $w(t)$ was also an arbitrary Riemann-integrable signal in $\mathcal{L}_2$ not identically zero, and hence

$$\sup_{w \in \mathcal{L}_2 \cap \mathcal{RI}} \frac{\int_{i=0}^{\infty} z_{K_\infty^c}(t)' z_{K_\infty^c}(t)}{\int_{i=0}^{\infty} w(t)' w(t)} \le \alpha.$$

□

For the decentralized sampled-data problem, the control gain is the same as for the state-feedback case (3.22) with the same design equation for X (3.23), as was the case for the two-rate problem. The other gains for the sampled-data problem are found as follows:

$$\begin{aligned}\sqrt{N}G_\delta K^d &= G_\delta \Upsilon^{-1}\left((D'_{GH}H_\delta + NG'_\delta X(I+TA_\delta)) + (D'_{GH}D_{BH} + TNG'_\delta XB_\delta)K^c\right) \\ &\longrightarrow \tfrac{1}{\alpha^2}(I - \tfrac{1}{\alpha^2}TM_6X)^{-1}[(M'_5 + M_6X(I+TA_\delta)) + (M'_9 + TM_6XB_\delta)K^c] \\ &= \widehat{GK^d},\end{aligned}$$

and

$$TNG_{\delta,C}\Upsilon^{-1}G'_{\delta,C} \longrightarrow \tfrac{1}{\alpha^2}T\begin{pmatrix}(I-\tfrac{1}{\alpha^2}TM_6X)^{-1}M_6 & \cdots & (I-\tfrac{1}{\alpha^2}TM_6X)^{-1}M_6 \\ \vdots & & \vdots \\ (I-\tfrac{1}{\alpha^2}TM_6X)^{-1}M_6 & \cdots & (I-\tfrac{1}{\alpha^2}TM_6X)^{-1}M_6\end{pmatrix},$$

$$G_{\delta,C}\Upsilon^{-1}D'_{GH}D_{BH} \longrightarrow \tfrac{1}{\alpha^2}\begin{pmatrix}(I-\tfrac{1}{\alpha^2}TM_6X)^{-1}M'_9 \\ \vdots \\ (I-\tfrac{1}{\alpha^2}TM_6X)^{-1}M'_9\end{pmatrix},$$

$$TNG_{\delta,C}\Upsilon^{-1}G'_\delta XB_\delta \longrightarrow \tfrac{1}{\alpha^2}\begin{pmatrix}(I-\tfrac{1}{\alpha^2}TM_6X)^{-1}M_6XB_\delta \\ \vdots \\ (I-\tfrac{1}{\alpha^2}TM_6X)^{-1}M_6XB_\delta\end{pmatrix}.$$

Thus, we obtain the following solution to Problem 3.2.3.

Theorem 3.4.2 *Given the decentralized continuous-time plant*

$$\dot{x} = A_c x + \sum_{i=1}^{p} B_c^i u_i + G_c w_0, \qquad x(0) = 0$$

$$y_i(kT) = C_i x(kT) + w_i(kT), \qquad k = 0,1,\ldots, \quad i \in \{1,2,\ldots,p\},$$

and performance variable

$$z = \begin{pmatrix} H_c x \\ u_1 \\ \vdots \\ u_p \end{pmatrix},$$

with unobservable modes of (A_c, H_c) in D_T^η, then the decentralized discrete-time controller

$$u_i(t) = K_i^c \xi_i(kT), \quad t \in [kT, (k+1)T); \qquad k = 0,1,2,\ldots,$$

for $i = 1, 2, ..., p$, where $\xi_i(kT)$ is the state of the full-state observer at the ith subsystem

$$\begin{aligned}\delta\xi_i = \{&A_\delta + B_\delta^i K_i^c + \textstyle\sum_{j\neq i} B_\delta^j K_j^c \\ &+ \tfrac{1}{\alpha^2}(I - \tfrac{1}{\alpha^2}TM_6X)^{-1}[(M_5' + M_6X(I+TA_\delta)) + (M_9' + TM_6XB_\delta)K^c]\}\xi_i(kT) \\ &+ K_i^o(y_i(kT) - C_i\xi_i(kT)), \qquad \xi_i(0) = 0,\end{aligned}$$

at time kT, and where K^c, K_D^o, $X > 0$, and $W > 0$ satisfy (3.22), (3.23), and

$$\begin{aligned}W = \tfrac{1}{\alpha^2}T &\begin{pmatrix} (I - \frac{1}{\alpha^2}TM_6X)^{-1}M_6 & \cdots & (I - \frac{1}{\alpha^2}TM_6X)^{-1}M_6 \\ \vdots & & \vdots \\ (I - \frac{1}{\alpha^2}TM_6X)^{-1}M_6 & \cdots & (I - \frac{1}{\alpha^2}TM_6X)^{-1}M_6 \end{pmatrix} + T\tfrac{1}{\alpha^2}K_D^oK_D^{o\prime} \\ &+ (I + TA_f - TK_D^oC_D) \\ &\quad \cdot((1-T\eta)^2W^{-1} - TK_D^{c\prime}[I + TB_\delta'XB_\delta + M_3 \\ &\qquad + \tfrac{1}{\alpha^2}(M_7 + TM_9XB_\delta + TB_\delta'XM_9' + T^2B_\delta'XM_6XB_\delta) \\ &\qquad + \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB_\delta'XM_6)\left(X^{-1} - \tfrac{1}{\alpha^2}TM_6\right)^{-1}(M_9 + TM_6XB_\delta)]K_D^c)^{-1} \\ &\quad \cdot(I + TA_f - TK_D^oC_D)',\end{aligned} \tag{3.24}$$

where

$$A_f := A_e - \frac{1}{\alpha^2}\begin{pmatrix} (I - \frac{1}{\alpha^2}TM_6X)^{-1}M_6XB_\delta \\ \vdots \\ (I - \frac{1}{\alpha^2}TM_6X)^{-1}M_6XB_\delta \end{pmatrix} K_D^c - \frac{1}{\alpha^2}\begin{pmatrix} (I - \frac{1}{\alpha^2}TM_6X)^{-1}M_9' \\ \vdots \\ (I - \frac{1}{\alpha^2}TM_6X)^{-1}M_9' \end{pmatrix} K_D^c,$$

such that

$$X^{-1} - \frac{1}{\alpha^2}\int_0^T e^{A_c(T-t)}G_cG_c'e^{A_c'(T-t)}\,dt > 0,$$

$$\begin{aligned}(1-T\eta)^2W^{-1} - TK_D^{c\prime}[I + TB_\delta'XB_\delta + M_3 + \tfrac{1}{\alpha^2}[M_7 + TM_9XB_\delta + TB_\delta'XM_9' + T^2B_\delta'XM_6XB_\delta] \\ + \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB_\delta'XM_6)\left(X^{-1} - \tfrac{1}{\alpha^2}TM_6\right)^{-1}(M_9 + TM_6XB_\delta)]K_D^c > 0,\end{aligned}$$

and

$$\begin{aligned}I + TB_\delta'XB_\delta + M_3 + \tfrac{1}{\alpha^2}[M_7 + TM_9XB_\delta + TB_\delta'XM_9' + T^2B_\delta'XM_6XB_\delta] \\ + \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB_\delta'XM_6)\left(X^{-1} - \tfrac{1}{\alpha^2}TM_6\right)^{-1}(M_9' + TM_6XB_\delta) > 0,\end{aligned}$$

and $A_e - K_D^oC_D$ has eigenvalues in D_T^η, where η is less than the degree of stability of the unobservable modes of (A_c, H_c), results in a stable closed-loop continuous-time system with Riemann H_∞ norm from w_e to z less than α.

Proof The proof of this is similar to that of Theorem 3.4.1 and is omitted. □

Note that (3.23) reduces, as $T \to 0$, to the continuous-time H_∞ state Riccati equation

$$0 = A_c'X + XA_c - XB_cB_c'X + \frac{1}{\alpha^2}XG_cG_c'X + H_c'H_c. \tag{3.25}$$

Using this fact, we find that the solution, for small T, is close to the continuous-time solution X. We make use of this in Section 3.6 to solve (3.23).

3.5 Reliable Sampled-data Decentralized Controller Design

In this section, decentralized controller designs are developed that are reliable to certain sets of sensor outages. Specifically, the designs are developed for sampled-data decentralized controllers that guarantee a given H_∞-norm bound on the continuous-time closed-loop system and closed-loop exponential stability for the system without failures and the system with any combinations of subsystem sensor outages in a prespecified set of subsystems. This is done by finding a reliable controller design for the two-rate system in Problem 3.2.4 and then taking the limit to obtain the sampled-data results.

Let $\Omega \subset \{1, 2, ..., p\}$ be the indices of the susceptible subsystems, and let $\omega \subseteq \Omega$ be the indices of the subsystems that actually experience failures.

Define the following matrices:

- $C_{\delta,\omega}$ as C_δ with the blocks not in ω set equal to zero
- $C_{\delta,\Omega}$ as C_δ with the blocks not in Ω set equal to zero
- $K_{D,\omega}^o$ as K_D^o with diagonal blocks not in ω set equal to zero
- $K_{D,\Omega}^o$ as K_D^o with diagonal blocks not in Ω set equal to zero
- $C_{\delta,\bar{\omega}} := C_\delta - C_{\delta,\omega}$ • $C_{\delta,\bar{\Omega}} := C_\delta - C_{\delta,\Omega}$
- $K_{D,\bar{\omega}}^o := K_D^o - K_{D,\omega}^o$.

The following theorem gives the reliable design for the two-rate system when subsystem sensor failures may occur (i.e., when $y_i = 0\ \forall i \in \omega \subseteq \Omega$).

Theorem 3.5.1 *For the decentralized system (3.10), (3.11), (3.12), with (A_δ, H_δ) detectable and with observer-based feedback $u_i = K_i^o\xi_i$, with observer (3.13), a sufficient condition to guarantee* *that the closed-loop plant is stable and that $\|T_{\tilde{w}_e\tilde{z}}\|_\infty \leq \alpha$ for all subsets of subsystem sensor failures $\omega \subseteq \Omega$ is*

$$\begin{aligned} K^c = -[&I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ &+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\ &\cdot [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\ &+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} T G_\delta' X (I + TA_\delta)\right)], \end{aligned}$$

$$K^d = \Upsilon^{-1} \left(\frac{1}{\sqrt{N}} D_{GH}'(H_\delta + D_{BH} K^c) + \sqrt{N} G_\delta' X (I + T(A_\delta + B_\delta K^c))\right),$$

where

$$\Upsilon := \alpha^2 I - TNG_\delta' X G_\delta - D_{GH}' D_{GH},$$

and K_D^o block diagonal, $X > 0$, and $W > 0$ satisfy

$$\begin{aligned} 0 = &A_\delta' X + X A_\delta + T A_\delta' X A_\delta + \tfrac{1}{N} H_\delta' H_\delta + \alpha^2 C_{\delta,\Omega}' C_{\delta,\Omega} \\ &+ \left(\tfrac{1}{\sqrt{N}} H_\delta' D_{GH} + \sqrt{N} (I + TA_\delta)' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right) \\ &- [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\ &+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]' \\ &\cdot [k_D' k_D + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ &+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\ &\cdot [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\ &+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)] \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} W = &TNG_{\delta,C} \Upsilon^{-1} G_{\delta,C}' + T \tfrac{1}{\alpha^2} K_D^o K_D^{o\prime} \\ &+ [I + T(A_e - G_{\delta,C} \Upsilon^{-1} [D_{GH}' D_{BH} + TNG_\delta' X B_\delta] K_D^c) - T K_D^o C_D] \\ &\cdot (W^{-1} - T K_D^{c\prime} [I + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ &\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)] K_D^c)^{-1} \\ &\cdot [I + T(A_e - G_{\delta,C} \Upsilon^{-1} [D_{GH}' D_{BH} + TNG_\delta' X B_\delta] K_D^c) - T K_D^o C_D]', \end{aligned}$$

such that the eigenvalues of $A_e - K_D^o C_D$ are in the stability region,

$$X^{-1} - TNG_\delta (\alpha^2 I - D_{GH}' D_{GH})^{-1} G_\delta' > 0,$$

$$\begin{aligned} W^{-1} - &T K_D^{c\prime} [I + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ &+ (\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta) \Upsilon^{-1} (\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta)] K_D^c > 0, \end{aligned}$$

and

$$|I + T(A_e - G_{\delta,C}\Upsilon^{-1}[D'_{GH}D_{BH} + TNG'_\delta X B_\delta]K^c_D) - TK^o_D C_D| \neq 0.$$

Proof See Appendix B.4. □

The only design equation that was modified to obtain reliability for this two-rate problem was the design equation for X. Thus, the only design equation in the sampled-data controller design that need be modified is the design equation for X. The modified design equation is thus

$$\begin{aligned}
0 = {} & A'_\delta X + XA_\delta + TA'_\delta XA_\delta + M_1 + \alpha^2 C'_\Omega C_\Omega \\
& + \tfrac{1}{\alpha^2}[M_4 + M_5 X(I + TA_\delta) + (I + TA_\delta)'XM'_5 + (I + TA_\delta)'XM_6 X(I + TA_\delta)] \\
& + \left(\tfrac{1}{\alpha^2}\right)^2 T(M_5 + (I + TA_\delta)'XM_6)\left(X^{-1} - \tfrac{1}{\alpha^2}TM_6\right)^{-1}(M'_5 + M_6 X(I + TA_\delta)) \\
& - \{B'_\delta X(I + TA_\delta) + M_2 + \tfrac{1}{\alpha^2}[M_8 + M_9 X(I + TA_\delta) + TB'_\delta XM'_5 + TB'_\delta XM_6 X(I + TA_\delta)] \\
& + \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB'_\delta XM_6)\left(X^{-1} - \tfrac{1}{\alpha^2}TM_6\right)^{-1}(M'_5 + M_6 X(I + TA_\delta))\}' \\
& \cdot \{I + TB'_\delta XB_\delta + M_3 + \tfrac{1}{\alpha^2}[M_7 + TM_9 XB_\delta + TB'_\delta XM'_9 + T^2 B'_\delta XM_6 XB_\delta] \\
& + \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB'_\delta XM_6)\left(X^{-1} - \tfrac{1}{\alpha^2}TM_6\right)^{-1}(M'_9 + TM_6 XB_\delta)\}^{-1} \\
& \cdot \{B'_\delta X(I + TA_\delta) + M_2 + \tfrac{1}{\alpha^2}[M_8 + M_9 X(I + TA_\delta) + TB'_\delta XM'_5 + TB'_\delta XM_6 X(I + TA_\delta)] \\
& + \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB'_\delta XM_6)\left(X^{-1} - \tfrac{1}{\alpha^2}TM_6\right)^{-1}(M'_5 + M_6 X(I + TA_\delta))\}.
\end{aligned} \tag{3.27}$$

3.6 Numerical Methods

In this section, numerical schemes for the solution of the sampled-data equations are discussed. First, iterative methods found for evaluating the design equations are presented. Then, the method of Van Loan [32], for evaluating integrals of exponentials by taking the exponential of a block matrix, is specialized to the integrals in the sampled-data design equations.

To find the solution to the design equation for X, first find the solution to the continuous-time H_∞ algebraic Riccati Equation (3.25). This can be done by evaluating the eigenvalues and eigenvectors of the associated Hamiltonian.

Next, for small T, iterate the update equation

$$\begin{aligned}
\delta X = {} & A_\delta' X + X A_\delta + T A_\delta' X A_\delta + M_1 \\
& + \tfrac{1}{\alpha^2}[M_4 + M_5 X(I+TA_\delta) + (I+TA_\delta)' X M_5' + (I+TA_\delta)' X M_6 X (I+TA_\delta)] \\
& + \left(\tfrac{1}{\alpha^2}\right)^2 T(M_5 + (I+TA_\delta)' X M_6) \left(X^{-1} - \tfrac{1}{\alpha^2} T M_6\right)^{-1} (M_5' + M_6 X(I+TA_\delta)) \\
& - \{B_\delta' X(I+TA_\delta) + M_2 + \tfrac{1}{\alpha^2}[M_8 + M_9 X(I+TA_\delta) + TB_\delta' X M_5' + TB_\delta' X M_6 X(I+TA_\delta)] \\
& + \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB_\delta' X M_6) \left(X^{-1} - \tfrac{1}{\alpha^2} T M_6\right)^{-1} (M_5' + M_6 X(I+TA_\delta))\}' \\
& \cdot \{I + TB_\delta' X B_\delta + M_3 + \tfrac{1}{\alpha^2}[M_7 + TM_9 X B_\delta + TB_\delta' X M_9' + T^2 B_\delta' X M_6 X B_\delta] \\
& + \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB_\delta' X M_6) \left(X^{-1} - \tfrac{1}{\alpha^2} T M_6\right)^{-1} (M_9' + TM_6 X B_\delta)\}^{-1} \\
& \cdot \{B_\delta' X(I+TA_\delta) + M_2 + \tfrac{1}{\alpha^2}[M_8 + M_9 X(I+TA_\delta) + TB_\delta' X M_5' + TB_\delta' X M_6 X(I+TA_\delta)] \\
& + \left(\tfrac{1}{\alpha^2}\right)^2 T(M_9 + TB_\delta' X M_6) \left(X^{-1} - \tfrac{1}{\alpha^2} T M_6\right)^{-1} (M_5' + M_6 X(I+TA_\delta))\}.
\end{aligned}$$

To evaluate the integrals M_1–M_9 in this equation, we apply the method developed by Van Loan. Let

$$R := \begin{pmatrix} -A_c' & H_c' H_c & 0 & 0 \\ 0 & A_c & G_c G_c' & 0 \\ 0 & 0 & -A_c' & H_c' H_c \\ 0 & 0 & 0 & A_c \end{pmatrix}.$$

Then, from [32],

$$e^{RT} = \begin{pmatrix} F_1(T) & G_1(T) & H_1(T) & K_1(T) \\ 0 & F_2(T) & G_2(T) & H_2(T) \\ 0 & 0 & F_3(T) & G_3(T) \\ 0 & 0 & 0 & F_4(T) \end{pmatrix},$$

where

$$F_1(T) = F_3(T) = e^{-A_c' T}, \quad F_2(T) = F_4(T) = e^{A_c T},$$

$$G_1(T) = G_3(T) = \int_0^T e^{-A_c'(T-t)} H_c' H_c e^{A_c t}\, dt, \quad G_2(T) = \int_0^T e^{A_c(T-t)} G_c G_c' e^{-A_c' t}\, dt,$$

$$H_1(T) = \int_0^T \int_0^t e^{-A_c'(T-t)} H_c' H_c e^{A_c(t-s)} G_c G_c' e^{-A_c' s}\, ds\, dt,$$

$$H_2(T) = \int_0^T \int_0^t e^{A_c(T-t)} G_c G_c' e^{-A_c'(t-s)} H_c' H_c e^{A_c s}\, ds\, dt,$$

$$K_1(T) = \int_0^T \int_0^s \int_0^t e^{-A_c'(T-s)} H_c' H_c e^{A_c(s-t)} G_c G_c' e^{-A_c'(t-\sigma)} H_c' H_c e^{A_c \sigma}\, d\sigma\, dt\, ds.$$

All the integrals can be expressed in terms of these five. (Note that

$$\int_0^T \int_t^T \int_t^T f(s,\sigma,t)\, ds\, d\sigma\, dt = \int_0^T \int_0^s \int_t^T f(s,\sigma,t)\, d\sigma\, dt\, ds.\ )$$

3.7 Example

In this section, we compare the performance of the H_∞-norm-bounding decentralized sampled-data controller designs on Example 2.8.1 with and without a degree of stability with that of the decentralized H_∞-norm-bounding discrete-time controller design, which was designed for the discretized plant.

For the comparison, the performance is measured in the following manner: The Hamiltonian matrices for the two-rate problems were found to be of the form (4.22), following the development for multirate systems to be described in Section 4.5.2. The limit was then taken, as $N \to \infty$, of the sequence of Hamiltonian matrices to obtain the matrix:

$$H := \begin{pmatrix} H_F + H_{GH} - TH_{GG}(I + TH'_F + TH'_{GH})^{-1}H_{HH} & H_{GG}(I + TH'_F + TH'_{GH})^{-1} \\ -(I + TH'_F + TH'_{GH})^{-1}H_{HH} & -(I + TH'_F + TH'_{GH})^{-1}(H'_F + H'_{GH}) \end{pmatrix}, \tag{3.28}$$

where

$$H_F := \begin{pmatrix} A_\delta + B_\delta K^c & B_\delta K^c_D \\ \widehat{GK^d} & \\ \vdots & \tilde{A}_e - K^o_D C_D \\ \widehat{GK^d} & \end{pmatrix},$$

$$\tilde{A}_e := \begin{pmatrix} A_\delta + B_\delta K^c + \widehat{GK^d} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_\delta + B_\delta K^c + \widehat{GK^d} \end{pmatrix} \tag{3.29}$$

$$+ \begin{pmatrix} -B^1_\delta K^c_1 & -B^2_\delta K^c_2 & \cdots & -B^p_\delta K^c_p \\ -B^1_\delta K^c_1 & \ddots & & \vdots \\ \vdots & & \ddots & -B^p_\delta K^c_p \\ -B^1_\delta K^c_1 & \cdots & -B^{p-1}_\delta K^c_{p-1} & -B^p_\delta K^c_p \end{pmatrix},$$

$$H_{GG} := \frac{1}{\alpha^2}\begin{pmatrix} M_6 & -M_6 & \cdots & M_6 \\ -M_6 & M_6 & \cdots & M_6 \\ \vdots & \vdots & & \vdots \\ -M_6 & M_6 & \cdots & M_6 \end{pmatrix} + \frac{1}{\alpha^2}\begin{pmatrix} 0 & 0 \\ 0 & K^o_D K^{o\prime}_D \end{pmatrix},$$

Table 3.1: Performance comparison for sampled-data and discrete-time decentralized controllers.

	Sampled-data		Discrete	
design α	α_H no degree of stability	α_H degree of stability	α_H	discrete H_∞ norm
8	3.48	3.28	3.50	3.50
4	3.01	2.86	3.05	3.05
2	1.988	1.925	1.993	1.994

$$H_{GH} := \frac{1}{\alpha^2}\begin{pmatrix} M_5' + M_9'K^c & M_9'K_D^c \\ -(M_5' + M_9'K^c) & -M_9'K_D^c \\ \vdots & \vdots \\ -(M_5' + M_9'K^c) & -M_9'K_D^c \end{pmatrix},$$

and

$$H_{HH} := \begin{pmatrix} M_1 + K^{c\prime}M_2 + M_2'K^c + K^{c\prime}(M_3+I)K^c & M_2'K_D^c + K^{c\prime}(M_3+I)K_D^c \\ K_D^{c\prime}M_2 + K_D^{c\prime}(M_3+I)K^c & K_D^{c\prime}(M_3+I)K_D^c \end{pmatrix}.$$

The smallest α of the desired precision is then found for which this matrix has no eigenvalues on the boundary of D_T. We denote the resulting α by α_H. For the discrete-time designs, let $\widehat{GK^d} = GK^d$ in (3.28) and (3.29). The comparison is given in Table 3.1. The sampled-data designs with a degree of stability were found by taking η to be 1/100th of the degree of stability of the continuous-time system. The sampling interval is $T = 0.01$.

3.8 Conclusions

Sampled-data and zero-order-hold state-feedback and decentralized controller designs were found to bound the continuous-time H_∞ norm of the closed-loop system. The decentralized controller design was modified to guarantee the H_∞-norm bound in spite of outages in a prespecified set of sensors. An example was worked to demonstrate that the resulting design performs slightly better than the standard discrete-time design. In the example, the performance was measured by the infimum of the α such that a certain matrix had no eigenvalues on the boundary of the stability region. Whether this corresponds to the actual continuous-time H_∞ norm of the system is left for future research.

CHAPTER 4

CONTROLLER DESIGNS FOR SYSTEMS WITH SENSORS AND ACTUATORS OPERATING AT MULTIPLE RATES

4.1 Introduction

In this chapter, H_∞-norm-bounding and sensor-outage reliable decentralized controller designs are developed for multirate digital control systems with sensors and actuators operating at different, rationally related, sampling and zero-order-hold rates. The norm for the multirate system is selected to correspond to the underlying continuous-time norm so that the design norm need not be redesigned for the multirate system.

A single-rate sampled-data representation of the multirate problem is obtained using the lifting techniques proposed by Meyer and Burrus [13] and generalized by Buescher and Grizzle [14], [15], and D. G. Meyer [16]. An approximation of the underlying continuous-time norm by means of Riemann sums is chosen to serve as the design norm for the multirate system.

The decentralized controller has observer-based controllers at each subsystem, as before. Each observer is chosen to have a predictive form, as proposed by Buescher [15], so that the controllers as applied to the multirate system remain causal.

The new, lifted, single-rate system has disturbance and control throughput terms at the measured output and a disturbance throughput term at the regulated output. The H_∞-norm-bounding controller design methods based on the generalized bounded real lemma of Chapter 3 are extended to this case. Because of difficulties in extending the reliable design to the case with a disturbance throughput term at the measured output, a slightly more conservative sensor-outage decentralized reliable design is developed and a bound is found for the H_∞-norm bound of that design as applied to the original multirate system.

The solutions to the basic and reliable multirate decentralized control problems are presented in the divided-difference operator formulation.

4.2 Lifting Multirate Control Problem to Single-rate Form

In this section, the multirate decentralized control problem is defined and a single-rate representation is obtained using the lifting technique of Meyer and Burrus [13], Buescher and Grizzle

[14], [15], and D. G. Meyer [29], [16]. The H_∞ norm is chosen for the multirate and lifted problem to approximate the underlying continuous-time H_∞ norm.

Suppose that the continuous-time system

$$\dot{x} = A_c x + \sum_{j=1}^{p} B_c^j u_j + G_c w_0, \quad x(0) = 0,$$

$$y_i = C_c^i x + w_i, \quad i \in \{1, 2, ..., p\},$$

$$z = \begin{pmatrix} H_c x \\ u_1 \\ \vdots \\ u_p \end{pmatrix}$$

is to be controlled with different, but rationally related, sampling rates for different sensors and actuators in the system.

Let T be the common longer period for which the overall system sampling is periodic. Suppose that u_i is piecewise constant on $t \in [m\frac{T}{k_i}, (m+1)\frac{T}{k_i})$ and y_i is sampled at $t = m\frac{T}{l_i}$, where $m = 0, 1, 2, ...$ and $i \in \{1, ..., p\}$. Assume also that w_i is piecewise constant on $t \in [m\frac{T}{l_i}, (m+1)\frac{T}{l_i})$, $m = 0, 1, 2, ...$.

Let N be the least common multiple of the k_i and l_j, $i, j \in \{1, ..., p\}$. Suppose that w_0 is piecewise constant on $t \in [m\frac{T}{N}, (m+1)\frac{T}{N})$ and that z is sampled at $m\frac{T}{N}$, $m = 0, 1, 2, ...$.

The evolution of the system from kT to $(k+1)T$ can be expressed in terms of the same state variable and extended control and output variables as

$$\delta x := \frac{x((k+1)T) - x(kT)}{T} = A_\delta x + \sum_{j=1}^{p} B_\delta^j u^j + G_\delta w^0, \quad x(0) = 0, \tag{4.1}$$

$$y^i = C_\delta^i x + \sum_{j=1}^{p} D_{BC_j}^i u^j + D_{GC}^i w^0 + w^i, \quad i \in \{1, ..., p\}, \tag{4.2}$$

where

$$u^j(kT) := \begin{pmatrix} u_j(kT) \\ u_j((k + \frac{1}{k_j})T) \\ \vdots \\ u_j((k + \frac{k_j - 1}{k_j})T) \end{pmatrix}, \quad w^0(kT) := \begin{pmatrix} w_0(kT) \\ w_0((k + \frac{1}{N})T) \\ \vdots \\ w_0((k + \frac{N-1}{N})T) \end{pmatrix},$$

$$y^i(kT) := \begin{pmatrix} y_i(kT) \\ y_i((k + \frac{1}{l_i})T) \\ \vdots \\ y_i((k + \frac{l_i - 1}{l_i})T) \end{pmatrix}, \quad w^i(kT) := \begin{pmatrix} w_i(kT) \\ w_i((k + \frac{1}{l_i})T) \\ \vdots \\ w_i((k + \frac{l_i - 1}{l_i})T) \end{pmatrix},$$

$$A_\delta := \frac{e^{A_cT} - I}{T}, \quad C_\delta^i := \begin{pmatrix} C_c^i \\ C_c^i e^{A_c(\frac{1}{l_i})T} \\ \vdots \\ C_c^i e^{A_c(\frac{l_i-1}{l_i})T} \end{pmatrix}$$

$$B_\delta^j := \begin{pmatrix} e^{A_c(\frac{k_j-1}{k_j})T}\frac{1}{T}\int_0^{\frac{T}{k_j}} e^{A_ct}\,dt\,B_c^j & \dots & e^{A_c(\frac{1}{k_j})T}\frac{1}{T}\int_0^{\frac{T}{k_j}} e^{A_ct}\,dt\,B_c^j & \frac{1}{T}\int_0^{\frac{T}{k_j}} e^{A_ct}\,dt\,B_c^j \end{pmatrix}$$

$$G_\delta := \begin{pmatrix} e^{A_c(\frac{N-1}{N})T}\frac{1}{T}\int_0^{\frac{T}{N}} e^{A_ct}\,dt\,G_c & \dots & e^{A_c(\frac{1}{N})T}\frac{1}{T}\int_0^{\frac{T}{N}} e^{A_ct}\,dt\,G_c & \frac{1}{T}\int_0^{\frac{T}{N}} e^{A_ct}\,dt\,G_c \end{pmatrix}$$

and $D_{BC_j}^i$ and D_{GC}^i are found as in [13] and [15]. For example, suppose that y_1 is sampled at $m\frac{T}{3}$ and u_2 is sampled at $m\frac{T}{2}$. Then

$$D_{BC_2}^1 = \begin{pmatrix} 0 & 0 \\ C_c^1\int_0^{2T/6} e^{A_ct}\,dt\,B_c^2 & 0 \\ C_c^1 e^{A_c(T/6)}\int_0^{3T/6} e^{A_ct}\,dt\,B_c^2 & C_c^1\int_0^{T/6} e^{A_ct}\,dt\,B_c^2 \end{pmatrix}.$$

All other cases are found similarly by expanding out the evolution equations resulting from the sampling.

Now consider, as we did for the sampled-data case, the H_∞ norm for the system with w_0 and z sampled at $t = m\frac{T}{N}$, $m = 0, 1, \dots$. We require that, in the limit as $N \to \infty$, the norm that we impose on the system correspond to the H_∞ norm of the underlying continuous-time system.

As before, we note that

$$\frac{\frac{T}{N}\sum_{i=0}^{\infty} z(i\frac{T}{N})'z(i\frac{T}{N})}{\frac{T}{N}\sum_{i=0}^{\infty} w_0(i\frac{T}{N})'w_0(i\frac{T}{N}) + \sum_{j=1}^{p}\frac{T}{N}\sum_{i=0}^{\infty} w_j(i\frac{T}{N})'w_j(i\frac{T}{N})}$$
$$\longrightarrow \frac{\int_0^\infty z(t)'z(t)\,dt}{\int_0^\infty w_0(t)'w_0(t)\,dt + \sum_{j=1}^{p}\int_0^\infty w_j(t)'w_j(t)\,dt}$$

if $N \to \infty$. Note that

$$\begin{aligned} \frac{T}{N}\sum_{i=0}^{\infty} z(i\tfrac{T}{N})'z(i\tfrac{T}{N}) &= \frac{T}{N}\sum_{i=0}^{\infty} x(i\tfrac{T}{N})'H_c'H_cx(i\tfrac{T}{N}) + \sum_{j=1}^{p}\frac{T}{N}\sum_{i=0}^{\infty} u_j(i\tfrac{T}{N})'u_j(i\tfrac{T}{N}) \\ &= \frac{T}{N}\sum_{i=0}^{\infty} x(i\tfrac{T}{N})'H_c'H_cx(i\tfrac{T}{N}) + \sum_{j=1}^{p}\frac{T}{k_j}\sum_{k=0}^{\infty} u^j(kT)'u^j(kT) \end{aligned}$$

and

$$\begin{aligned} &\frac{T}{N}\sum_{i=0}^{\infty} w_0(i\tfrac{T}{N})'w_0(i\tfrac{T}{N}) + \sum_{j=1}^{p}\frac{T}{N}\sum_{i=0}^{\infty} w_j(i\tfrac{T}{N})'w_j(i\tfrac{T}{N}) \\ &\quad = \frac{T}{N}\sum_{k=0}^{\infty} w^0(kT)'w^0(kT) + \sum_{j=1}^{p}\frac{T}{l_j}\sum_{k=0}^{\infty} w^j(kT)'w^j(kT), \end{aligned}$$

and

$$\begin{pmatrix} H_cx(kT) \\ H_cx(kT + (\frac{1}{N})T) \\ \vdots \\ H_cx(kT + (\frac{N-1}{N})T) \end{pmatrix} = H_\delta x(kT) + \sum_{j=1}^{p} D_{BH_j}u^j(kT) + D_{GH}w^0(kT),$$

where

$$H_\delta := \begin{pmatrix} H_c \\ H_c e^{A_c(\frac{1}{N})T} \\ \vdots \\ H_c e^{A_c(\frac{N-1}{N})T} \end{pmatrix}, \quad D_{GH} := \begin{pmatrix} 0 & \cdots & 0 \\ H_c \int_0^{\frac{T}{N}} e^{A_c t}\, dt\, G_c & \ddots & \vdots \\ \vdots & & \ddots \\ H_c e^{A_c(\frac{N-2}{N})T} \int_0^{\frac{T}{N}} e^{A_c t}\, dt\, G_c & \cdots \quad H_c \int_0^{\frac{T}{N}} e^{A_c t}\, dt\, G_c & 0 \end{pmatrix},$$

and D_{BH_j} is the appropriate matrix, as can be found in the expansion. Thus,

$$\sup_{w_0 \in \ell_2} \frac{\frac{T}{N}\sum_{i=0}^{\infty} z(i\frac{T}{N})' z(i\frac{T}{N})}{\frac{T}{N}\sum_{i=0}^{\infty} w_0(i\frac{T}{N})' w_0(i\frac{T}{N}) + \sum_{j=1}^{p} \frac{T}{N}\sum_{i=0}^{\infty} w_j(i\frac{T}{l_j})' w_j(i\frac{T}{l_j})} = \sup_{w^0 \in \ell_2} \frac{T\sum_{k=0}^{\infty} \tilde{z}(kT)'\tilde{z}(kT)}{T\sum_{k=0}^{\infty} \tilde{w}_e(kT)'\tilde{w}_e(kT)},$$

where

$$\tilde{w}^0(kT) := \frac{1}{\sqrt{N}} w^0(kT), \quad \tilde{w}^i(kT) := \frac{1}{\sqrt{l_i}} w^i(kT),$$

$$\tilde{w}(kT) := \begin{pmatrix} \tilde{w}^1(kT) \\ \vdots \\ \tilde{w}^p(kT) \end{pmatrix}, \quad \tilde{w}_e(kT) := \begin{pmatrix} \tilde{w}^0(kT) \\ \tilde{w}(kT) \end{pmatrix},$$

and

$$\tilde{z}(kT) := \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta x(kT) + \sum_{j=1}^{p} D_{BH_j} u^j(kT)) + D_{GH}\tilde{w}^0(kT) \\ \frac{1}{\sqrt{k_1}} u^1(kT) \\ \vdots \\ \frac{1}{\sqrt{k_p}} u^p(kT) \end{pmatrix}.$$

Then the equivalent single-rate problem that we would like to solve is the following:

Problem 4.2.1 *Given the discrete-time decentralized plant*

$$\delta x = A_\delta x + \sum_{j=1}^{p} B_\delta^j u^j + \sqrt{N} G_\delta \tilde{w}^0, \quad x(0) = 0, \tag{4.3}$$

$$y^i = C_\delta^i x + \sum_{j=1}^{p} D_{BC_j}^i u^j + \sqrt{N} D_{GC}^i \tilde{w}^0 + \sqrt{l_i}\tilde{w}^i \tag{4.4}$$

with the performance-output variable

$$\tilde{z} = \begin{pmatrix} \frac{1}{\sqrt{N}}\left(H_\delta x + \sum_{j=1}^{p} D_{BH_j} u^j\right) + D_{GH}\tilde{w}^0 \\ \frac{1}{\sqrt{k_1}} u^1(kT) \\ \vdots \\ \frac{1}{\sqrt{k_p}} u^p(kT) \end{pmatrix}, \tag{4.5}$$

design a decentralized discrete-time controller of the form $u_i = K_i^c \xi_i$, *where*

$$\delta\xi_i = \left(A_\delta + B_\delta^i K_i^c + \sum_{j\neq i} B_\delta^j K_j^c + \sqrt{N} G_\delta K^d\right)\xi_i + K_i^o\left(y^i - \left(C_\delta^i + \sum_{j=1}^{p} D_{BC_j}^i K_j^c + \sqrt{N} D_{GC}^i K^d\right)\xi_i\right), \tag{4.6}$$

$$\xi_i(0) = 0,$$

for which the closed-loop system is stable and has H_∞ *norm from* $\tilde{w}_e$ *to* $\tilde{z}$ *less than a given value* α.

Note that the observer form is predictive. It uses the same form of estimates for control inputs at other channels and the disturbance w^0 as before.

4.3 Multirate Controller Design

In this section, the discrete-time decentralized controller design that solves Problem 4.2.1 is presented.

Define the following composite matrices:

$$B_\delta := (B_\delta^1\ B_\delta^2 \cdots B_\delta^p),\qquad D_{BH} := (D_{BH_1}\ D_{BH_2} \cdots D_{BH_p}),\qquad D_{BC}^i := (D_{BC_1}^i\ D_{BC_2}^i \cdots D_{BC_p}^i),$$

$$G_{\delta,C} := \begin{pmatrix} G_\delta \\ \vdots \\ G_\delta \end{pmatrix},\qquad D_{BC} := \begin{pmatrix} D_{BC}^1 \\ \vdots \\ D_{BC}^p \end{pmatrix},\qquad D_{BC,D} := \begin{pmatrix} D_{BC}^1 & 0 & \cdots & 0 \\ 0 & D_{BC}^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & D_{BC}^p \end{pmatrix},$$

$$D_{GC} := \begin{pmatrix} D_{GC}^1 \\ \vdots \\ D_{GC}^p \end{pmatrix},\qquad D_{GC,D} := \begin{pmatrix} D_{GC}^1 & 0 & \cdots & 0 \\ 0 & D_{GC}^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & D_{GC}^p \end{pmatrix},$$

$$K^c := \begin{pmatrix} K_1^c \\ \vdots \\ K_p^c \end{pmatrix},\qquad K_D^c := \begin{pmatrix} K_1^c & 0 & \cdots & 0 \\ 0 & K_2^c & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K_p^c \end{pmatrix},\qquad K_{C,D}^c := \begin{pmatrix} K^c & 0 & \cdots & 0 \\ 0 & K^c & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K^c \end{pmatrix},$$

$$K_D^d := \begin{pmatrix} K^d & 0 & \cdots & 0 \\ 0 & K^d & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K^d \end{pmatrix},\qquad K_D^o := \begin{pmatrix} K_1^o & 0 & \cdots & 0 \\ 0 & K_2^o & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K_p^o \end{pmatrix},\qquad C_D := \begin{pmatrix} C_\delta^1 & 0 & \cdots & 0 \\ 0 & C_\delta^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & C_\delta^p \end{pmatrix},$$

$$k_D := \begin{pmatrix} \frac{1}{\sqrt{k_1}}I & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{k_2}}I & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\sqrt{k_p}}I \end{pmatrix}, \qquad l_D := \begin{pmatrix} \sqrt{l_1}I & 0 & \cdots & 0 \\ 0 & \sqrt{l_2}I & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sqrt{l_p}I \end{pmatrix},$$

and

$$\begin{aligned} A_e \; := & \begin{pmatrix} A_\delta + B_\delta K^c + \sqrt{N}G_\delta K^d & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_\delta + B_\delta K^c + \sqrt{N}G_\delta K^d \end{pmatrix} \\ & + \begin{pmatrix} -B_\delta^1 K_1^c & -B_\delta^2 K_2^c & \cdots & -B_\delta^p K_p^c \\ -B_\delta^1 K_1^c & \ddots & & \vdots \\ \vdots & & \ddots & -B_\delta^p K_p^c \\ -B_\delta^1 K_1^c & \cdots & -B_\delta^{p-1} K_{p-1}^c & -B_\delta^p K_p^c \end{pmatrix}. \end{aligned}$$

Then the following theorem gives sufficient conditions for the multirate decentralized controller design.

Theorem 4.3.1 *For the decentralized system (4.3), (4.4), (4.5), with (A_δ, H_δ) detectable and with observer-based feedback $u_i = K_i^c \xi_i$, with observer (4.6), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{\tilde{w}_e \tilde{z}}\|_\infty \leq \alpha$ is*

$$\begin{aligned} K^c = & -[k_D' k_D + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ & + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\ & \cdot [B_\delta' X(I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\ & + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X(I + T A_\delta)\right)], \end{aligned} \tag{4.7}$$

$$K^d = \Upsilon^{-1} \left(\frac{1}{\sqrt{N}} D_{GH}'(H_\delta + D_{BH} K^c) + \sqrt{N} G_\delta' X(I + T(A_\delta + B_\delta K^c))\right), \tag{4.8}$$

where

$$\Upsilon := \alpha^2 I - T N G_\delta' X G_\delta - D_{GH}' D_{GH},$$

and K_D^o block diagonal, $X > 0$, and $W > 0$ satisfy

$$
\begin{aligned}
0 = {} & A_\delta' X + X A_\delta + T A_\delta' X A_\delta + \tfrac{1}{N} H_\delta' H_\delta \\
& \quad + \left(\tfrac{1}{\sqrt{N}} H_\delta' D_{GH} + \sqrt{N}(I + T A_\delta)' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right) \\
& - [B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\
& \quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right)]' \\
& \cdot [k_D' k_D + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\
& \quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
& \cdot [B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\
& \quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right)]
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
W = {} & T N (G_{\delta,C} - K_D^o D_{GC}) \Upsilon^{-1} (G_{\delta,C} - K_D^o D_{GC})' + T \tfrac{1}{\alpha^2} K_D^o (l_D l_D') K_D^{o\prime} \\
& + [I + T(A_e - (G_{\delta,C} - K_D^o D_{GC}) \Upsilon^{-1} [D_{GH}' D_{BH} + T N G_\delta' X B_\delta] K_D^c) \\
& \qquad - T K_D^o (C_D + D_{BC,D} K_{C,D}^c + D_{GC,D} K_D^d - D_{BC} K_D^c)] \\
& \cdot (W^{-1} - T K_D^{o\prime} [k_D' k_D + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\
& \qquad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)] K_D^c)^{-1} \\
& \cdot [I + T(A_e - (G_{\delta,C} - K_D^o D_{GC}) \Upsilon^{-1} [D_{GH}' D_{BH} + T N G_\delta' X B_\delta] K_D^c) \\
& \qquad - T K_D^o (C_D + D_{BC,D} K_{C,D}^c + D_{GC,D} K_D^d - D_{BC} K_D^c)]',
\end{aligned}
\tag{4.10}
$$

such that the eigenvalues of $A_e - K_D^o (C_D + D_{BC,D} K_{C,D}^c + D_{GC,D} K_D^d)$ are in the stability region, $\Upsilon > 0$,

$$
\begin{aligned}
& W^{-1} - T K_D^{o\prime} [k_D' k_D + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\
& \qquad + (\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta) \Upsilon^{-1} (\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta)] K_D^c > 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& |I + T(A_e - (G_{\delta,C} - K_D^o D_{GC}) \Upsilon^{-1} [D_{GH}' D_{BH} + T N G_\delta' X B_\delta] K_D^c) \\
& \qquad - T K_D^o (C_D + D_{BC,D} K_{C,D}^c + D_{GC,D} K_D^d - D_{BC} K_D^c)| \neq 0.
\end{aligned}
$$

The choice of $K_D^o = \alpha^2 W_D (C_D + D_{BC,D} K_{C,D}^c + D_{GC,D} K_D^d - D_{BC} K_D^c)' (l_D l_D')^{-1}$, where W_D is the block diagonal part of W, results in one solution.

Proof sketch:

Let $e_i := \xi_i - x$. Then the closed-loop system matrices can be expressed as

$$
F_e = \begin{pmatrix} A_\delta + B_\delta K^c & B_\delta K_D^c \\ \sqrt{N}(G_{\delta,C} - K_D^o D_{GC}) K^d & A_e - K_D^o C_{D+} \end{pmatrix}, \qquad G_e = \begin{pmatrix} \sqrt{N} G_\delta & 0 \\ -\sqrt{N}(G_{\delta,C} - K_D^o D_{GC}) & K_D^o l_D \end{pmatrix},
$$

$$H_e = \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta + D_{BH}K^c) & \frac{1}{\sqrt{N}}D_{BH}K^c_D \\ k_D K^c & k_D K^c_D \end{pmatrix}, \qquad E_e = \begin{pmatrix} D_{GH} & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$C_{D+} := C_D + D_{BC,D}K^c_{C,D} + D_{GC,D}K^d_D - D_{BC}K^c_D.$$

As before, let $X_e = \left(\begin{smallmatrix} X & 0 \\ 0 & X_1 \end{smallmatrix}\right)$. Applying the conditions of the generalized bounded real lemma, as in Chapter 3, the conditions in the theorem are derived. □

4.4 Reliable Multirate Controller Design

For the reliable case, the disturbance throughput term in the measured output equation creates difficulties in the development of the design. Consequentially, a bound is found on the conservatism of the design found by disregarding that term, and such a reliable design is developed.

To accommodate the throughput term in $\tilde{w}^0$, consider the composite noise $v := D_{GC}\tilde{w}^0 + \frac{1}{\sqrt{N}} l_D \tilde{w}$. Since $\tilde{w}$ is independent of $\tilde{w}^0$ and since the sum of two signals in ℓ_2 is also in ℓ_2, the noise v is free to take on the full range of values, and thus we can base the design for disturbance rejection on the closed-loop system from $\left(\begin{smallmatrix} \tilde{w}^0 \\ v \end{smallmatrix}\right)$ to z. Then, we obtain

$$\alpha \geq \frac{\|z\|_2}{\left\| \begin{pmatrix} \tilde{w}^0 \\ v \end{pmatrix} \right\|_2} = \frac{\|z\|_2}{\left\| \begin{pmatrix} I & 0 \\ D_{GC} & \frac{1}{\sqrt{N}} l_D \end{pmatrix} \begin{pmatrix} \tilde{w}^0 \\ \tilde{w} \end{pmatrix} \right\|_2}$$

$$\geq \frac{\|z\|_2}{\left\| \begin{pmatrix} I & 0 \\ D_{GC} & \frac{1}{\sqrt{N}} l_D \end{pmatrix} \right\|_2 \left\| \begin{pmatrix} \tilde{w}^0 \\ \tilde{w} \end{pmatrix} \right\|_2} = \beta \frac{\|z\|_2}{\left\| \begin{pmatrix} \tilde{w}^0 \\ \tilde{w} \end{pmatrix} \right\|_2},$$

where

$$\beta := \frac{1}{\left\| \begin{pmatrix} I & 0 \\ D_{GC} & \frac{1}{\sqrt{N}} l_D \end{pmatrix} \right\|_2} = \frac{1}{\lambda_{max}^{\frac{1}{2}} \begin{pmatrix} I + D'_{GC}D_{GC} & \frac{1}{\sqrt{N}} D'_{GC} l_D \\ \frac{1}{\sqrt{N}} l'_D D_{GC} & \frac{1}{N} l'_D l_D \end{pmatrix}}.$$

For small T, D_{GC} is small and $\frac{1}{N} l'_D l_D$ has eigenvalues no greater than 1, and thus $\frac{1}{\beta}$ is close to 1. The resulting (conservative) H_∞-norm bound for the original system is $\frac{1}{\beta}\alpha$.

The results for the reliable multirate case with the control throughput term in the measured-output equation are different from those for the single-rate reliable case. If a sensor outage occurs, terms relating not only to the sensor, but also to the actuator, go to zero.

First, the decentralized multirate controller design problem and solution are restated for this, more conservative, problem. Then, the decentralized multirate design is modified to produce a design that is reliable to subsystem sensor outages of the type $y_i = 0$ for each susceptible subsystem i.

The following is the single-rate controller design problem with the composite disturbance considered:

Problem 4.4.1 *Given the discrete-time decentralized plant*

$$\delta x = A_\delta x + \sum_{j=1}^{p} B_\delta^j u^j + \sqrt{N} G_\delta \tilde{w}^0, \quad x(0) = 0, \tag{4.11}$$

$$y^i = C_\delta^i x + \sum_{j=1}^{p} D_{BC_j}^i u^j + \sqrt{N} v^i \tag{4.12}$$

with the performance-output variable

$$\tilde{z} = \begin{pmatrix} \frac{1}{\sqrt{N}} \left(H_\delta x + \sum_{j=1}^{p} D_{BH_j} u^j \right) + D_{GH} \tilde{w}^0 \\ \frac{1}{\sqrt{k_1}} u^1(kT) \\ \vdots \\ \frac{1}{\sqrt{k_p}} u^p(kT) \end{pmatrix}, \tag{4.13}$$

design a decentralized discrete-time controller of the form $u_i = K_i^c \xi_i$, where

$$\delta \xi_i = \left(A_\delta + B_\delta^i K_i^c + \sum_{j \neq i} B_\delta^j K_j^c + \sqrt{N} G_\delta K^d \right) \xi_i + K_i^o \left(y^i - \left(C_\delta^i + \sum_{j=1}^{p} D_{BC_j}^i K_j^c \right) \xi_i \right), \quad \xi_i(0) = 0, \tag{4.14}$$

for which the closed-loop system is stable and has H_∞ norm from $\binom{\tilde{w}^0}{v}$ to $\tilde{z}$ less than a given value α.

The H_∞-norm-bounding decentralized controller design that satisfies Problem 4.4.1 is as follows:

Theorem 4.4.1 *For the decentralized system (4.11), (4.12), (4.13), with (A_δ, H_δ) detectable and with observer-based feedback $u_i = K_i^c \xi_i$, with observer (4.14), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{\binom{\tilde{w}^0}{v}\tilde{z}}\|_\infty \le \alpha$ is*

$$\begin{aligned} K^c = -[&k_D' k_D + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ &+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta \right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta \right)]^{-1} \\ \cdot [&B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\ &+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta \right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta) \right)], \end{aligned}$$

$$K^d = \Upsilon^{-1}\left(\frac{1}{\sqrt{N}}D'_{GH}(H_\delta + D_{BH}K^c) + \sqrt{N}G'_\delta X(I + T(A_\delta + B_\delta K^c))\right),$$

where

$$\Upsilon := \alpha^2 I - TNG'_\delta X G_\delta - D'_{GH}D_{GH},$$

and K^o_D block diagonal, $X > 0$, and $W > 0$ satisfy

$$\begin{aligned}
0 = {} & A'_\delta X + XA_\delta + TA'_\delta XA_\delta + \tfrac{1}{N}H'_\delta H_\delta \\
& \quad + \left(\tfrac{1}{\sqrt{N}}H'_\delta D_{GH} + \sqrt{N}(I + TA_\delta)'XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}H_\delta + \sqrt{N}G'_\delta X(I + TA_\delta)\right) \\
& - [B'_\delta X(I + TA_\delta) + \tfrac{1}{N}D'_{BH}H_\delta \\
& \quad + \left(\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}H_\delta + \sqrt{N}G'_\delta X(I + TA_\delta)\right)]' \\
& \cdot [k'_D k_D + TB'_\delta XB_\delta + \tfrac{1}{N}D'_{BH}D_{BH} \\
& \quad + \left(\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_\delta XB_\delta\right)]^{-1} \\
& \cdot [B'_\delta X(I + TA_\delta) + \tfrac{1}{N}D'_{BH}H_\delta \\
& \quad + \left(\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}H_\delta + \sqrt{N}G'_\delta X(I + TA_\delta)\right)]
\end{aligned}$$

and

$$\begin{aligned}
W = {} & TNG_{\delta,C}\Upsilon^{-1}G'_{\delta,C} + TN\tfrac{1}{\alpha^2}K^o_D K^{o\prime}_D \\
& + [I + T(A_e - G_{\delta,C}\Upsilon^{-1}[D'_{GH}D_{BH} + TNG'_\delta XB_\delta]K^c_D) \\
& \qquad - TK^o_D(C_D + D_{BC,D}K^c_{C,D} - D_{BC}K^c_D)] \\
& \cdot (W^{-1} - TK^{o\prime}_D[k'_D k_D + TB'_\delta XB_\delta + \tfrac{1}{N}D'_{BH}D_{BH} \\
& \qquad + \left(\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_\delta XB_\delta\right)]K^c_D)^{-1} \\
& \cdot [I + T(A_e - G_{\delta,C}\Upsilon^{-1}[D'_{GH}D_{BH} + TNG'_\delta XB_\delta]K^c_D) \\
& \qquad - TK^o_D(C_D + D_{BC,D}K^c_{C,D} - D_{BC}K^c_D)]',
\end{aligned}$$

such that the eigenvalues of $A_e - K^o_D(C_D + D_{BC,D}K^c_{C,D})$ are in the stability region, $\Upsilon > 0$,

$$\begin{aligned}
& W^{-1} - TK^{o\prime}_D[k'_D k_D + TB'_\delta XB_\delta + \tfrac{1}{N}D'_{BH}D_{BH} \\
& \qquad + (\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta)\Upsilon^{-1}(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_\delta XB_\delta)]K^c_D > 0,
\end{aligned}$$

and

$$\begin{aligned}
& |I + T(A_e - G_{\delta,C}\Upsilon^{-1}[D'_{GH}D_{BH} + TNG'_\delta XB_\delta]K^c_D) \\
& \qquad - TK^o_D(C_D + D_{BC,D}K^c_{C,D} - D_{BC}K^c_D)| \neq 0.
\end{aligned}$$

The choice of $K^o_D = \alpha^2\frac{1}{N}W_D(C_D + D_{BC,D}K^c_{C,D} - D_{BC}K^c_D)'$, where W_D is the block diagonal part of W, results in one solution. Because of the "simplification" made, a different starting point is required for the iterative numerical algorithm to converge, as discussed in Section 4.5.

Now, this decentralized problem is modified to guarantee reliability to subsystem sensor outages. Let $\Omega \subset \{1, 2, ..., p\}$ be the indices of the susceptible subsystems, and let $\omega \subseteq \Omega$ be the indices of the subsystems that actually experience failures.

Define the following matrices:

- $C_{\delta,\omega}$ as C_δ with the blocks not in ω set equal to zero
- $C_{\delta,\Omega}$ as C_δ with the blocks not in Ω set equal to zero
- $K^o_{D,\omega}$ as K^o_D with diagonal blocks not in ω set equal to zero
- $K^o_{D,\Omega}$ as K^o_D with diagonal blocks not in Ω set equal to zero
- $D_{BC,\omega}$ as D_{BC} with rows of blocks not in ω set equal to zero
- $D_{BC,\Omega}$ as D_{BC} with rows of blocks not in Ω set equal to zero
- $C_{\delta,\bar{\omega}} := C_\delta - C_{\delta,\omega}$ • $C_{\delta,\bar{\Omega}} := C_\delta - C_{\delta,\Omega}$ • $K^o_{D,\bar{\omega}} := K^o_D - K^o_{D,\omega}$
- $D_{BC,\bar{\omega}} := D_{BC} - D_{BC,\omega}$ • $D_{BC,\bar{\Omega}} := D_{BC} - D_{BC,\Omega}$.

The following theorem gives the reliable controller design for the case in which subsystem sensor failures may occur (i.e., when $y_i = 0 \; \forall i \in \omega \subseteq \Omega$).

Theorem 4.4.2 *For the decentralized system (4.11), (4.12), (4.13), with (A_δ, H_δ) detectable and with observer-based feedback $u_i = K^c_i \xi_i$, with observer (4.14), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{\binom{\bar{w}^o}{v}\bar{z}}\|_\infty \leq \alpha$ for all subsets of subsystem sensor failures $\omega \subseteq \Omega$ is*

$$\begin{aligned} K^c = -[&k'_D k_D + TB'_\delta X B_\delta + \tfrac{1}{N} D'_{BH} D_{BH} + \tfrac{1}{N}\alpha^2 D'_{BC,\Omega} D_{BC,\Omega} \\ &+ \left(\tfrac{1}{\sqrt{N}} D'_{BH} D_{GH} + \sqrt{N} T B'_\delta X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D'_{GH} D_{BH} + \sqrt{N} T G'_\delta X B_\delta\right)]^{-1} \\ \cdot [&B'_\delta X (I + TA_\delta) + \tfrac{1}{N} D'_{BH} H_\delta + \tfrac{1}{N}\alpha^2 D'_{BC,\Omega} C_{\delta,\Omega} \\ &+ \left(\tfrac{1}{\sqrt{N}} D'_{BH} D_{GH} + \sqrt{N} T B'_\delta X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D'_{GH} H_\delta + \sqrt{N} G'_\delta X (I + TA_\delta)\right)], \end{aligned} \tag{4.15}$$

$$K^d = \Upsilon^{-1}\left(\frac{1}{\sqrt{N}} D'_{GH}(H_\delta + D_{BH}K^c) + \sqrt{N} G'_\delta X (I + T(A_\delta + B_\delta K^c))\right),$$

where

$$\Upsilon := \alpha^2 I - TNG'_\delta X G_\delta - D'_{GH} D_{GH},$$

and K_D^o block diagonal, $X > 0$, and $W > 0$ satisfy

$$
\begin{aligned}
0 = {} & A_\delta' X + X A_\delta + T A_\delta' X A_\delta + \tfrac{1}{N} H_\delta' H_\delta + \tfrac{1}{N}\alpha^2 C_{\delta,\Omega}' C_{\delta,\Omega} \\
& + \left(\tfrac{1}{\sqrt{N}} H_\delta' D_{GH} + \sqrt{N}(I + TA_\delta)' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right) \\
& - [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \tfrac{1}{N}\alpha^2 D_{BC,\Omega}' C_{\delta,\Omega} \\
& \quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]' \\
& \cdot [k_D' k_D + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} + \tfrac{1}{N}\alpha^2 D_{BC,\Omega}' D_{BC,\Omega} \\
& \quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
& \cdot [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \tfrac{1}{N}\alpha^2 D_{BC,\Omega}' C_{\delta,\Omega} \\
& \quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]
\end{aligned}
\tag{4.16}
$$

and

$$
\begin{aligned}
W = {} & T N G_{\delta,C} \Upsilon^{-1} G_{\delta,C}' + T N \tfrac{1}{\alpha^2} K_D^o K_D^{o\prime} \\
& + [I + T(A_e - G_{\delta,C} \Upsilon^{-1} [D_{GH}' D_{BH} + T N G_\delta' X B_\delta] K_D^c) \\
& \qquad - T K_D^o (C_D + D_{BC,D} K_{C,D}^c - D_{BC} K_D^c)] \\
& \cdot (W^{-1} - T K_D^{o\prime} [k_D' k_D + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} + \tfrac{1}{N}\alpha^2 D_{BC,\Omega}' D_{BC,\Omega} \\
& \qquad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)] K_D^c)^{-1} \\
& \cdot [I + T(A_e - G_{\delta,C} \Upsilon^{-1} [D_{GH}' D_{BH} + T N G_\delta' X B_\delta] K_D^c) \\
& \qquad - T K_D^o (C_D + D_{BC,D} K_{C,D}^c - D_{BC} K_D^c)]',
\end{aligned}
\tag{4.17}
$$

such that the eigenvalues of $A_e - K_D^o(C_D + D_{BC,D} K_{C,D}^c)$ are in the stability region, $\Upsilon > 0$,

$$
\begin{aligned}
& W^{-1} - T K_D^{o\prime} [k_D' k_D + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} + \tfrac{1}{N}\alpha^2 D_{BC,\Omega}' D_{BC,\Omega} \\
& \qquad + (\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta) \Upsilon^{-1} (\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta)] K_D^c > 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& |I + T(A_e - G_{\delta,C} \Upsilon^{-1} [D_{GH}' D_{BH} + T N G_\delta' X B_\delta] K_D^c) \\
& \qquad - T K_D^o (C_D + D_{BC,D} K_{C,D}^c - D_{BC} K_D^c)| \neq 0.
\end{aligned}
$$

4.5 Numerical Issues

4.5.1 Modifications to numerical update schemes for multirate equations

The multirate designs require a slightly modified numerical update scheme for the solution of the decentralized design equations for W. In the single-rate update schemes of Section 2.6, K_D^o was chosen to be $\alpha^2 W_D C_D'$ and the iteration was consistently started at the solution to the continuous-time algebraic Riccati-like Equation (A.4).

In developing the multirate design equations, B_δ, C_δ, et cetera, were expanded in the lifting, and the weightings k_D and l_D were introduced into the design equations, as determined by the derivations based on Lemma 3.3.1. Appropriate weightings must be found for both K_D^o and for the continuous-time design equation to find the starting point for the iteration. This is accomplished as follows.

The form of the design equations from each theorem is found when $T = 0$. The solutions of the design equations are close to the solution of these "continuous-time" design equations for small T. However, these design equations are not necessarily the design equations for the continuous-time controller designs. In particular, they are different from the continuous-time controller design equations for Theorems 4.4.1 and 4.4.2, due to the "simplification" that removed the throughput term in w_0 from the measured-output equation.

The form for K_D^o is chosen in each case to make it possible to solve the design equation for $T = 0$, as done by Veillette in [25].

For Theorem 4.3.1, the design equations reduce as follows: (4.7) and (4.8) reduce to

$$K^c = -\begin{pmatrix} B_c^{1\prime} \\ \vdots \\ B_c^{1\prime} \\ B_c^{2\prime} \\ \vdots \\ B_c^{2\prime} \\ B_c^{p\prime} \\ \vdots \\ B_c^{p\prime} \end{pmatrix} X, \qquad K^d = \frac{1}{\sqrt{N}}\frac{1}{\alpha^2}\begin{pmatrix} G_c' \\ \vdots \\ G_c' \end{pmatrix} X, \tag{4.18}$$

where there are k_i blocks of $B_c^{i\prime}$, $i = 1, ..., p$, and N blocks of G_c'. Consequently, (4.9) becomes

$$0 = A_c'X + XA_c + \frac{1}{\alpha^2}XG_cG_c'X - XB_cB_c'X + H_c'H_c, \tag{4.19}$$

and (4.10) becomes

$$\begin{aligned} 0 = A_eW + WA_e' + \tfrac{1}{\alpha^2}G_CG_C' + WX_DB_DB_D'X_DW + \tfrac{1}{\alpha^2}K_D^o(l_Dl_D')K_D^{o\prime} \\ - K_D^oC_{De}W - WC_{De}'K_D^{o\prime}, \end{aligned}$$

where now G_C and B_D refer to the continuous-time block matrices and

$$C_{De} := \begin{pmatrix} C_c^1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ C_c^1 & 0 & \cdots & \cdots & 0 \\ 0 & C_c^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & C_c^2 & 0 & \cdots & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & \cdots & \cdots & 0 & C_c^p \\ \vdots & & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & C_c^p \end{pmatrix}.$$

To make this last equation solvable, K_D^o is chosen to be $K_D^o = \alpha^2 W_D C_{De}'(l_D l_D')^{-1}$. Then, the initial value of W is chosen as the solution to

$$\begin{aligned} 0 = {} & A_e W + W A_e' + \tfrac{1}{\alpha^2} G_C G_C' + W X_D B_D B_D' X_D W - \alpha^2 W C_{De}'(l_D l_D')^{-1} C_{De} W \\ & + \alpha^2 (W - W_D) C_{De}'(l_D l_D')^{-1} C_{De}(W - W_D). \end{aligned}$$

Note finally that $C_{De}'(l_D l_D')^{-1} C_{De} = C_D' C_D$, where now C_D refers to the continuous-time block matrix. Thus, this is the continuous-time design Equation (A.4). Hence, in this case, the choice of the initial point for the iteration is the solution to the continuous-time design Equation (A.4) if the choice $K_D^o = \alpha^2 W_D C_{De}'(l_D l_D')^{-1}$ is made for the observer gain when $T = 0$. However, this choice of observer gain is the value of

$$K_D^o = \alpha^2 W_D (C_D + D_{BC,D} K_{C,D}^c + D_{GC,D} K_D^d - D_{BC} K_D^c)'(l_D l_D')^{-1} \tag{4.20}$$

when $T = 0$. The choice is made of this observer gain (4.20) form for Theorem 4.3.1.

For Theorem 4.4.1, the design equations reduce to (4.18) and (4.19). The design equation for W reduces, when $T = 0$, to

$$\begin{aligned} 0 = {} & A_e W + W A_e' + \tfrac{1}{\alpha^2} G_C G_C' + W X_D B_D B_D' X_D W + N \tfrac{1}{\alpha^2} K_D^o K_D^{o\prime} \\ & - K_D^o C_{De} W - W C_{De}' K_D^{o\prime}. \end{aligned}$$

If $K_D^o = \frac{1}{N} \alpha^2 W_D C_{De}'$, then this becomes

$$\begin{aligned} 0 = {} & A_e W + W A_e' + \tfrac{1}{\alpha^2} G_C G_C' + W X_D B_D B_D' X_D W - \tfrac{1}{N} \alpha^2 W C_D' C_D W \\ & + \tfrac{1}{N} \alpha^2 (W - W_D) C_{De}'(l_D l_D')^{-1} C_{De}(W - W_D), \end{aligned}$$

which may be solved iteratively. The observer gain for Theorem 4.4.1 is chosen as

$$K_D^o = \frac{1}{N}\alpha^2 W_D(C_D + D_{BC,D}K^c_{C,D} - D_{BC}K^c_D)',$$

which reduces for $T = 0$ to $K_D^o = \frac{1}{N}\alpha^2 W_D C'_{De}$.

For Theorem 4.4.2, the gains again reduce to (4.18) when $T = 0$, but the design equation for X becomes

$$0 = A_c'X + XA_c + \frac{1}{\alpha^2}XG_cG_c'X - XB_cB_c'X + H_c'H_c + \alpha^2 \sum_{i\in\Omega} \frac{l_i}{N} C_c^{i\prime} C_c^i. \tag{4.21}$$

The design equation for W reduces as for Theorem 4.4.1. The solution to this has to be calculated based on the solution X to (4.21). Thus, the observer gain is of the form

$$K_D^o = \frac{1}{N}\alpha^2 W_D(C_D + D_{BC,D}K^c_{C,D} - D_{BC}K^c_D)'$$

for Theorem 4.4.2, but the initial condition is different, which is reasonable since the reliable theorems produce controllers that perform quite differently from the other theorems.

4.5.2 Hamiltonian method for computing the norm of a system with a disturbance throughput term in the regulated-output equation

The H_∞ norm of the designs in Chapter 2 were calculated, after the design gains were computed, by finding the smallest value of α for which the appropriate Hamiltonian matrix had no stability region boundary eigenvalues. However, the closed-loop systems for the multirate designs, in this chapter and in Chapter 3, have disturbance throughput terms in the regulated-output equations. The form of the Hamiltonian used to calculate the H_∞ norms of the designs in Chapter 2 was for the system with no throughput terms.

The appropriate Hamiltonian for use for the systems with throughput terms is obtained as follows. For $T \neq 0$,

$$0 = F'X+XF+TF'XF+H'H+(H'E+(I+TF)'XG)(\alpha^2 I-TG'XG-E'E)^{-1}(E'H+G'X(I+TF))$$

$$\iff 0 = [(I+TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']XL^{-1}[(I+TF) + TG(\alpha^2 I - E'E)^{-1}E'H]$$
$$- X + TH'(I - \tfrac{1}{\alpha^2}EE')^{-1}H,$$

where $L := I - TG(\alpha^2 I - E'E)^{-1}G'X$,

$$\iff 0 = X[(I+TF) + TG(\alpha^2 I - E'E)^{-1}E'H]$$
$$- (I - TXG(\alpha^2 I - E'E)^{-1}G')[(I+TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}X$$
$$+ (I - TXG(\alpha^2 I - E'E)^{-1}G')[(I+TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}TH'(I - \tfrac{1}{\alpha^2}EE')^{-1}H$$

$$
\begin{aligned}
\iff 0 = X[&I + TF + TG(\alpha^2 I - E'E)^{-1}E'H \\
&- TG(\alpha^2 I - E'E)^{-1}G'[(I + TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}TH'(I - \tfrac{1}{\alpha^2}EE')^{-1}H] \\
&- [(I+TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}X \\
&+ [(I+TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}TH'(I - \tfrac{1}{\alpha^2}EE')^{-1}H] \\
&+ X[TG(\alpha^2 I - E'E)^{-1}G'[(I+TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}]X.
\end{aligned}
$$

Thus, the difference-equation form of the Hamiltonian matrix is

$$
M_q = \begin{pmatrix} M_{11,q} & M_{12,q} \\ M_{21,q} & M_{22,q} \end{pmatrix},
$$

where

$$
\begin{aligned}
M_{11,q} &= I + TF + TG(\alpha^2 I - E'E)^{-1}E'H \\
&\quad - TG(\alpha^2 I - E'E)^{-1}G'[(I + TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}TH'(I - \tfrac{1}{\alpha^2}EE')^{-1}H, \\
M_{12,q} &= TG(\alpha^2 I - E'E)^{-1}G'[(I + TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}, \\
M_{21,q} &= -[(I + TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}TH'(I - \frac{1}{\alpha^2}EE')^{-1}H, \\
M_{22,q} &= [(I + TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}.
\end{aligned}
$$

Subtracting I and dividing by T, we obtain the Hamiltonian matrix for the system with throughput terms, in the divided-difference form:

$$
M_\delta = \begin{pmatrix} M_{11,\delta} & M_{12,\delta} \\ M_{21,\delta} & M_{22,\delta} \end{pmatrix}, \tag{4.22}
$$

where

$$
\begin{aligned}
M_{11,\delta} &= F + G(\alpha^2 I - E'E)^{-1}E'H \\
&\quad - TG(\alpha^2 I - E'E)^{-1}G'[(I + TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}H'(I - \tfrac{1}{\alpha^2}EE')^{-1}H, \\
M_{12,\delta} &= G(\alpha^2 I - E'E)^{-1}G'[(I + TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}, \\
M_{21,\delta} &= -[(I + TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}H'(I - \frac{1}{\alpha^2}EE')^{-1}H, \\
M_{22,\delta} &= -[(I + TF)' + TH'E(\alpha^2 I - E'E)^{-1}G']^{-1}(F' + H'E(\alpha^2 I - E'E)^{-1}G').
\end{aligned}
$$

4.6 Examples

In this section, an example of the multirate designs is presented. The example presented is Example 2.8.1 with $u_1(t) = u_1(mT/4)$ for $t \in [mT/4,(m+1)T/4)$, $u_2(t) = u_2(mT/2)$ for $t \in [mT/2,(m+1)T/2)$, y_1 sampled at $t = mT/3$ and y_2 sampled at $mT/2$, for $m = 0,1,\ldots$.

The resulting lifted system matrices are

$$A_\delta = \frac{e^{A_cT}-I}{T}, \qquad C_\delta^1 = \begin{pmatrix} C_c^1 \\ C_c^1 e^{A_c(T/3)} \\ C_c^1 e^{A_c(2T/3)} \end{pmatrix}, \qquad C_\delta^2 = \begin{pmatrix} C_c^2 \\ C_c^2 e^{A_c(T/2)} \end{pmatrix},$$

$$B_\delta^1 = \begin{pmatrix} e^{A_c(3T/4)}\frac{1}{T}\int_0^{T/4} e^{A_ct}\,dt\,B_c^1 & \cdots & e^{A_c(T/4)}\frac{1}{T}\int_0^{T/4} e^{A_ct}\,dt\,B_c^1 & \frac{1}{T}\int_0^{T/4} e^{A_ct}\,dt\,B_c^1 \end{pmatrix},$$

$$B_\delta^2 = \begin{pmatrix} e^{A_c(T/2)}\frac{1}{T}\int_0^{T/2} e^{A_ct}\,dt\,B_c^2 & \frac{1}{T}\int_0^{T/2} e^{A_ct}\,dt\,B_c^2 \end{pmatrix},$$

$$G_\delta = \begin{pmatrix} e^{A_c(11T/12)}\frac{1}{T}\int_0^{T/12} e^{A_ct}\,dt\,G_c & \cdots & e^{A_c(T/12)}\frac{1}{T}\int_0^{T/12} e^{A_ct}\,dt\,G_c & \frac{1}{T}\int_0^{T/12} e^{A_ct}\,dt\,G_c \end{pmatrix},$$

$$D_{BC_1}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ C_c^1 e^{A_c(T/12)}\int_0^{T/4} e^{A_ct}\,dt\,B_c^1 & C_c^1\int_0^{T/12} e^{A_ct}\,dt\,B_c^1 & 0 & 0 \\ C_c^1 e^{A_c(5T/12)}\int_0^{T/4} e^{A_ct}\,dt\,B_c^1 & C_c^1 e^{A_c(T/6)}\int_0^{T/4} e^{A_ct}\,dt\,B_c^1 & C_c^1\int_0^{T/6} e^{A_ct}\,dt\,B_c^1 & 0 \end{pmatrix},$$

$$D_{BC_1}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ C_c^2 e^{A_c(T/4)}\int_0^{T/4} e^{A_ct}\,dt\,B_c^1 & C_c^2\int_0^{T/4} e^{A_ct}\,dt\,B_c^1 & 0 & 0 \end{pmatrix},$$

$$D_{BC_2}^1 = \begin{pmatrix} 0 & 0 \\ C_c^1\int_0^{T/3} e^{A_ct}\,dt\,B_c^2 & 0 \\ C_c^1 e^{A_c(T/6)}\int_0^{T/2} e^{A_ct}\,dt\,B_c^2 & C_c^1\int_0^{T/6} e^{A_ct}\,dt\,B_c^2 \end{pmatrix},$$

$$D_{BC_2}^2 = \begin{pmatrix} 0 & 0 \\ C_c^2\int_0^{T/2} e^{A_ct}\,dt\,B_c^2 & 0 \end{pmatrix}, \qquad H_\delta = \begin{pmatrix} H_c \\ H_c e^{A_c(T/12)} \\ \vdots \\ H_c e^{A_c(11T/12)} \end{pmatrix},$$

$$D_{GC}^1 = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 \\ C_c^1 e^{A_c(3T/12)}\int_0^{T/12} e^{A_ct}\,dt\,G_c & \cdots & C_c^1\int_0^{T/12} e^{A_ct}\,dt\,G_c & 0 & 0 & 0 \\ C_c^1 e^{A_c(7T/12)}\int_0^{T/12} e^{A_ct}\,dt\,G_c & \cdots & C_c^1 e^{A_c(4T/12)}\int_0^{T/12} e^{A_ct}\,dt\,G_c & \cdots & C_c^1\int_0^{T/12} e^{A_ct}\,dt\,G_c & 0 \end{pmatrix},$$

$$D_{GC}^2 = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ C_c^2 e^{A_c(5T/12)}\int_0^{T/12} e^{A_ct}\,dt\,G_c & \cdots & C_c^2\int_0^{T/12} e^{A_ct}\,dt\,G_c & 0 \end{pmatrix},$$

$$D_{GH} := \begin{pmatrix} 0 & & \cdots & 0 \\ H_c \int_0^{T/12} e^{A_c t}\,dt\,G_c & \ddots & & \vdots \\ \vdots & & \ddots & \\ H_c e^{A_c(10T/12)} \int_0^{T/12} e^{A_c t}\,dt\,G_c & \cdots & H_c \int_0^{T/12} e^{A_c t}\,dt\,G_c & 0 \end{pmatrix},$$

$$D_{BH_1} = \begin{pmatrix} \Phi_1 & 0 & 0 & 0 \\ \Phi_2 & \Phi_1 & 0 & 0 \\ \Phi_3 & \Phi_2 & \Phi_1 & 0 \\ \Phi_4 & \Phi_3 & \Phi_2 & \Phi_1 \end{pmatrix}, \qquad D_{BH_2} = \begin{pmatrix} \Phi_5 & 0 \\ \Phi_6 & \Phi_5 \end{pmatrix},$$

where

$$\Phi_1 = \begin{pmatrix} 0 \\ H_c \int_0^{T/12} e^{A_c t}\,dt\,B_c^1 \\ H_c \int_0^{2T/12} e^{A_c t}\,dt\,B_c^1 \end{pmatrix}, \qquad \Phi_2 = \begin{pmatrix} H_c \int_0^{3T/12} e^{A_c t}\,dt\,B_c^1 \\ H_c e^{A_c(T/12)} \int_0^{3T/12} e^{A_c t}\,dt\,B_c^1 \\ H_c e^{A_c(2T/12)} \int_0^{3T/12} e^{A_c t}\,dt\,B_c^1 \end{pmatrix},$$

$$\Phi_3 = \begin{pmatrix} H_c e^{A_c(3T/12)} \int_0^{3T/12} e^{A_c t}\,dt\,B_c^1 \\ H_c e^{A_c(4T/12)} \int_0^{3T/12} e^{A_c t}\,dt\,B_c^1 \\ H_c e^{A_c(5T/12)} \int_0^{3T/12} e^{A_c t}\,dt\,B_c^1 \end{pmatrix}, \qquad \Phi_4 = \begin{pmatrix} H_c e^{A_c(6T/12)} \int_0^{3T/12} e^{A_c t}\,dt\,B_c^1 \\ H_c e^{A_c(7T/12)} \int_0^{3T/12} e^{A_c t}\,dt\,B_c^1 \\ H_c e^{A_c(8T/12)} \int_0^{3T/12} e^{A_c t}\,dt\,B_c^1 \end{pmatrix},$$

$$\Phi_5 = \begin{pmatrix} H_c \int_0^{T/12} e^{A_c t}\,dt\,B_c^2 \\ H_c \int_0^{2T/12} e^{A_c t}\,dt\,B_c^2 \\ H_c \int_0^{3T/12} e^{A_c t}\,dt\,B_c^2 \\ H_c \int_0^{4T/12} e^{A_c t}\,dt\,B_c^2 \\ H_c \int_0^{5T/12} e^{A_c t}\,dt\,B_c^2 \end{pmatrix}, \qquad \Phi_6 = \begin{pmatrix} H_c \int_0^{6T/12} e^{A_c t}\,dt\,B_c^2 \\ H_c e^{A_c(T/12)} \int_0^{T/2} e^{A_c t}\,dt\,B_c^2 \\ H_c e^{A_c(2T/12)} \int_0^{T/2} e^{A_c t}\,dt\,B_c^2 \\ H_c e^{A_c(3T/12)} \int_0^{T/2} e^{A_c t}\,dt\,B_c^2 \\ H_c e^{A_c(4T/12)} \int_0^{T/2} e^{A_c t}\,dt\,B_c^2 \\ H_c e^{A_c(5T/12)} \int_0^{T/2} e^{A_c t}\,dt\,B_c^2 \end{pmatrix},$$

and

$$k_D := \begin{pmatrix} \frac{1}{\sqrt{4}}I & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{4}}I & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{4}}I & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{4}}I & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}I & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}I \end{pmatrix}, \qquad l_D := \begin{pmatrix} \sqrt{3}I & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3}I & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3}I & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}I & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}I \end{pmatrix},$$

First, it is demonstrated in Table 4.1 that the resulting norms from the multirate design of Theorem 4.3.1 are close to the norms from the single-rate design of Theorem 2.4.1 and that the norms from the more conservative multirate design of Theorem 4.4.1 using the same design values are higher. In this case, the multirate decentralized controller could be found for a lower value than that of the single-rate decentralized controller. The minimum design value for the centralized observer-based controller was 1.7. Recall that the decentralized controller is slightly conservative in the way that it was formed. Therefore, nothing can be concluded from this about the "optimal" value of the problem.

Table 4.1: H_∞ norms for varying α. Decentralized single-rate and multirate controllers.

	Single-rate	Multirate	Conservative Multirate
α	$\|T\|_\infty$	$\|T\|_\infty$	$\|T\|_\infty$
20	3.64	3.64	4.52
16	3.63	3.63	4.47
12	3.59	3.59	4.36
8	3.50	3.50	4.07
4.33	3.13	3.13	3.15
4	3.05	3.05	no solution
2	1.994	1.992	no solution
1.82	no solution	1.76	no solution

Next, in Table 4.2, performance in the presence of sensor outages is compared for the multirate decentralized controller design of Theorem 4.3.1, the more conservative multirate decentralized controller design of Theorem 4.4.1, and the reliable decentralized multirate controller design of Theorem 4.4.2.

The norms are measured from the remaining disturbances entering the system to the regulated output for the full multirate system. The multirate system matrices for the system with sensor outages $y_i = 0$ for all $i \in \omega$ were computed as

$$\bar{F}_e = \begin{pmatrix} A_\delta + B_\delta K^c & B_\delta K^c_D \\ \sqrt{N}(G_{\delta,C} - K^o_D D_{GC})K^d - K^o_{D,\omega}(C_{\delta,\omega} + D_{BC,\omega}K^c) & A_e - K^o_D C_{D+} - K^o_{D,\omega} D_{BC,\omega} K^c_D \end{pmatrix},$$

$$\bar{G}_e = \begin{pmatrix} \sqrt{N} G_\delta & 0 \\ -\sqrt{N}(G_{\delta,C} - K^o_D D_{GC}) - \sqrt{N} K^o_{D,\omega} D_{GC,\omega} & K^o_{D,\bar{\omega}} l_{D,\bar{\omega}} \end{pmatrix},$$

$$\bar{H}_e = \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta + D_{BH}K^c) & \frac{1}{\sqrt{N}} D_{BH} K^c_D \\ k_D K^c & k_D K^c_D \end{pmatrix},$$

and

$$\bar{E}_e = \begin{pmatrix} D_{GH} & 0 \\ 0 & 0 \end{pmatrix},$$

where $C_{D+} := C_D + D_{BC,D}K^c_{C,D} + D_{GC,D}K^d_D - D_{BC}K^c_D$, $l_{D,\bar{\omega}}$ is l_D with the diagonal blocks with indices in ω set to zero, and $D_{GC,\omega}$ is D_{GC} with rows of blocks not in ω set to zero.

Table 4.2: H_∞ norms in the presence of sensor outages for multirate decentralized designs.

	Multirate			Conservative Multirate			Reliable (to $y_1 = 0$)		
α	no failure	$y_1 = 0$	$y_2 = 0$	no failure	$y_1 = 0$	$y_2 = 0$	no failure	$y_1 = 0$	$y_2 = 0$
29	3.66	unstable	5.37	4.57	unstable	7.32	7.15	8.86	7.20
26	3.66	unstable	5.37	4.56	unstable	7.29	7.65	8.68	7.77

Again, as for the single-rate designs of Chapter 2, the closed-loop system becomes unstable if $y_1 = 0$ when the H_∞-norm-bounding designs are applied. The reliable controller design (for outages of y_1) stabilizes the system for any single sensor outage. Because the multirate reliable design is more conservative, the norm bounds resulting from the multirate reliable controller are higher than in the single-rate case in Table 2.2.

4.7 Conclusions

In this chapter, H_∞-norm-bounding and sensor-outage reliable decentralized controller designs were developed for control systems with sensors and actuators operating at different, rationally related, sampling and zero-order-hold rates. A simplified problem was solved in the reliable controller design case adding to the conservatism of the design. A bound was found on the conservatism of the design. The numerical scheme for finding the solution to the decentralized design equations of Chapter 2 was modified to apply to the design equations resulting in this chapter. An example was then worked, and a new Hamiltonian matrix was derived to determine the H_∞ norm of the resulting closed-loop system, which has a disturbance throughput term in the regulated-output equation.

CHAPTER 5

CONCLUSIONS

This thesis presented a unified approach to designing H_∞-norm-bounding and reliable decentralized controllers for the continuous-time, discrete-time, sampled-data, and multirate control problems.

In Chapter 2, a unified approach for continuous- and discrete-time systems was presented for the H_∞-norm-bounding and reliable controller designs of [1] and [3] for state-feedback controllers and centralized and decentralized observer-based controllers. The reliable designs guaranteed stability and a performance-norm bound despite sensor or actuator outages in a prespecified set. In addition, the decentralized design was modified to guarantee a prespecified degree of stability for the closed-loop system. A unified form of the bounded-real lemma in [1] and [3] was proved to derive the controller designs. Numerical techniques were presented for the solution of the design equations arising in the discrete-time decentralized problems.

In Chapter 3, a new state-space approach to sampled-data H_∞-norm-bounding controller design was presented. Explicit design equations were derived for decentralized H_∞-norm-bounding controllers by finding the limit of a sequence of two-rate decentralized controller design problems, each with the disturbance and regulated-output variables measured at multiple instances between sampling periods. A more general bounded real lemma was found to accommodate the form of the two-rate problems with a disturbance throughput term in the regulated-output equation. The decentralized design was then modified to guarantee stability and a norm bound despite sensor outages in a prespecified set.

In Chapter 4, a decentralized controller design for multirate sampled-data digital control systems was found that guarantees stability and a prespecified H_∞-norm bound for the closed-loop system. This involved lifting the multirate system to a single-rate form and then extending some of the results of Chapter 2 to systems with throughput terms in the measured-output, as well as in the regulated-output, equation. A more conservative design was found, which was then extended to guarantee stability and an H_∞-norm bound despite sensor outages in any subset of a preselected set of subsystems sensor connections. The H_∞ norm of the closed-loop system was calculated in the example using a Hamiltonian matrix that was derived for systems with throughput terms in the output equation.

The design norms for all of the problems addressed were related back to the H_∞ norm of the underlying continuous-time system, obviating the need for the designer to "redesign" the controller in each case and facilitating the solution of the single-rate and multirate discrete-time design equations. An example was worked to compare the performance obtained from each type of controller design and to demonstrate that the solution methods proposed are effective.

More work should be done in the area of reliable control to make it viable for application to real control systems.

The reliable controller designs herein were derived for the infinite-horizon problem with zero initial conditions. In actual systems, outages sometimes occur during operation and the designer will be concerned with the transient response of the system to these outages. An "H_∞-norm"-bounding problem with nonzero initial conditions has been considered in [33], [34], for a new H_∞-like performance measure with weightings on the initial and final conditions imbedded in the norm. This can be approached using an extension of the bounded real lemma, for the design of reliable controllers for systems with nonzero initial conditions and, also, finite time horizon.

The reliable designs in this thesis guarantee performance and stability for outages in subsets of a limited set of the sensors or actuators. One would prefer designs that would perform well in spite of *any* single outage. Alternatively, one might like to optimize, in some sense, the combined use of reliable control and fault detection in a system.

Other designs that guarantee reliability to sensor and actuator outages could be found. A development of these and a comparison with the reliable designs herein would be of interest. Also, sensors and actuators can fail in other manners. For instance, sensor or actuator signals could be attenuated without being entirely lost, a constant bias term could be added to the signals, or the signal might be lost and the sensor or actuator noise remain. Designs that guarantee performance despite these other failure modes would be desirable. Reliable designs that bound other system norms would also be useful, especially to solve the reliable problem against bias terms in the sensor or actuator signals.

Further, research could be done into how redundancy should be optimally added to the system to enhance system reliability. The designs developed herein, or other reliable designs, could be used in developing analysis tools for determining where sensors and actuators should be placed in the system.

APPENDIX A

PROOFS OF LEMMAS AND THEOREMS IN CHAPTER 2

The proofs of the lemmas and theorems in Chapter 2 are presented here.

A.1 Proofs of the Bounded Real Lemmas

First, the proof of Lemma 2.2.1 is presented.

Lemma Consider a linear system T_{wz} with a detectable realization

$$\rho x = Fx + Gw, \qquad x(0) = 0,$$

$$z = Hx.$$

If there exist a real, symmetric matrix $X \geq 0$ and a real $\alpha > 0$ such that

(i) $F'XL^{-1} + XL^{-1}F + TF'XL^{-1}F + \frac{1}{\alpha^2}XGG'XL^{-1} + H'H \leq 0$

(ii) $\alpha^2 I - TG'XG > 0$

where $L := I - T\frac{1}{\alpha^2}GG'X$, then

(a) the eigenvalues of F lie in D_T, the stability region for sampling interval T

(b) $\|T_{wz}\|_\infty \leq \alpha$.

Proof

(a): Let $v \neq 0$ be an eigenvector of F satisfying $Fv = \lambda v$. From (i),

$$\begin{aligned}
0 &\geq v^*(F'XL^{-1} + XL^{-1}F + TF'XL^{-1}F + \tfrac{1}{\alpha^2}XGG'XL^{-1} + H'H)v \\
&= \left(2\Re(\lambda) + T|\lambda|^2\right) v^*(XL^{-1})v + \tfrac{1}{\alpha^2}v^*XGG'XL^{-1}v + v^*H'Hv \\
&= \left(2\Re(\lambda) + T|\lambda|^2\right) v^*(X + TXG(\alpha^2 I - TG'XG)^{-1}G'X)v \\
&\quad + v^*XG(\alpha^2 I - TG'XG)^{-1}G'Xv + v^*H'Hv.
\end{aligned}$$

From (ii), each term is positive semidefinite, with the possible exception of the first. Thus, the first term is negative semidefinite:

$$\left(2\Re(\lambda) + T|\lambda|^2\right) v^*(X + TXG(\alpha^2 I - TG'XG)^{-1}G'X)v \leq 0.$$

If

$$\left(2\Re(\lambda) + T|\lambda|^2\right) v^*(X + TXG(\alpha^2 I - TG'XG)^{-1}G'X)v < 0,$$

then

$$v^*(X + TXG(\alpha^2 I - TG'XG)^{-1}G'X)v > 0$$

and

$$2\Re(\lambda) + T|\lambda|^2 < 0.$$

In that case, λ is in the stability region. If

$$\left(2\Re(\lambda) + T|\lambda|^2\right) v^*(X + TXG(\alpha^2 I - TG'XG)^{-1}G'X)v = 0,$$

then all of the terms must be zero, and thus, in particular, $Hv = 0$. Since (F, H) is detectable, $\lambda \in D_T$, which proves (a).

(b): Showing that

$$\|T_{wz}\|_\infty \leq \alpha$$

is equivalent to showing that

$$V := \|z\|_2{}^2 - \alpha^2 \|w\|_2{}^2 \leq 0, \quad \forall w \in \mathcal{L}_2.$$

First note that

$$\mathcal{S}_{t=0}^{\infty} \frac{x(t+T)'Xx(t+T) - x(t)'Xx(t)}{T} \, dt = -x(0)'Xx(0) =: 0.$$

(In the continuous-time case, the integral is equal to

$$\lim_{t \to \infty} x(t)'Xx(t) - x(0)'Xx(0),$$

the first term of which is 0 since, by (a), the system is stable. In the discrete-time case, the partial sums are

$$x(kT)'Xx(kT) - x(0)'Xx(0).$$

These converge to $-x(0)'Xx(0)$ since the system is stable.)

Now, we show that $V \leq 0$.

$$\begin{aligned}
V &= \|z\|_2{}^2 - \alpha^2\|w\|_2{}^2 - x(0)'Xx(0) \\
&= \mathcal{S}_{t=0}^{\infty}[z(t)'z(t) - \alpha^2 w(t)'w(t) \\
&\qquad + \tfrac{1}{T}(x(t+T)'Xx(t+T) - x(t)'Xx(t))]\, dt \\
&= \mathcal{S}_{t=0}^{\infty}[x(t)'\left(H'H + \tfrac{1}{T}(I+TF)'X(I+TF) - \tfrac{1}{T}X\right)x(t) \\
&\qquad + 2x(t)'\left((I+TF)'XG\right)w(t) + w(t)'\left(TG'XG - \alpha^2 I\right)w(t)]\, dt \\
&= \mathcal{S}_{t=0}^{\infty}[(x(t)'\left(H'H + F'X + XF + TF'XF\right))x(t) \\
&\qquad + 2x(t)'\left((I+TF)'XG\right)w(t) + w(t)'\left(TG'XG - \alpha^2 I\right)w(t)]\, dt.
\end{aligned}$$

But, by (i),

$$\begin{aligned}
&F'XL^{-1} + XL^{-1}F + TF'XL^{-1}F + \tfrac{1}{\alpha^2}XGG'XL^{-1} + H'H \\
&= H'H + F'X + XF + TF'XF \\
&\quad + (I+TF)'XG(\alpha^2 I - TG'XG)^{-1}G'X(I+TF) \\
&\leq 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
V &\leq \mathcal{S}_{t=0}^{\infty}(x(t)'\left(-(I+TF)'XG(\alpha^2 I - TG'XG)^{-1}G'X(I+TF)\right)x(t) \\
&\qquad + 2x(t)'\left((I+TF)'XG\right)w(t) + w(t)'\left(-(\alpha^2 I - TG'XG)\right)w(t))\, dt \\
&= -\mathcal{S}_{t=0}^{\infty}\left(w(t) - (\alpha^2 I - TG'XG)^{-1}G'X(I+TF)x(t)\right)' \\
&\qquad \cdot (\alpha^2 I - TG'XG)\left(w(t) - (\alpha^2 I - TG'XG)^{-1}G'X(I+TF)x(t)\right)\, dt \\
&\leq 0, \qquad \text{by (ii).}
\end{aligned}$$

□

Next, the proof is presented for Lemma 2.5.1, which is similar to the proof of Lemma 2.2.1, modified to guarantee that the system has a prescribed degree of stability, as defined in Definition 2.5.1.

Lemma Consider a linear system T_{wz} with a realization

$$\rho x = Fx + Gw, \qquad x(0) = 0,$$

$$z = Hx$$

with all unobservable modes of F in D_T^η. If there exist a real, symmetric matrix $X \geq 0$ and a real $\alpha > 0$ such that

(i) $F'XL^{-1} + XL^{-1}F + TF'XL^{-1}F + \frac{1}{\alpha^2}XGG'XL^{-1} + H'H \leq -(2 - T\eta)\eta X$

(ii) $\alpha^2 I - TG'XG > 0$

where $L := I - T\frac{1}{\alpha^2}GG'X$, then

(a) the eigenvalues of F lie in D_T^η

(b) $\|T_{wz}\|_\infty \leq \alpha$.

Proof

(a): Let $v \neq 0$ be an eigenvector of F satisfying $Fv = \lambda v$. From (i),

$$0 \geq \left((2 - T\eta)\eta + 2\Re(\lambda) + T|\lambda|^2\right) v^*(XL^{-1})v + (1 - T(2 - T\eta)\eta)\frac{1}{\alpha^2}v^*XGG'XL^{-1}v + v^*H'Hv.$$

Since $1 - T(2 - T\eta)\eta = (1 - T\eta)^2 > 0$, each term is positive semidefinite, as in the proof of Lemma 2.2.1, with the possible exception of the first. Thus, the first term is negative semidefinite:

$$\left((2 - T\eta)\eta + 2\Re(\lambda) + T|\lambda|^2\right) v^*(X + TXG(\alpha^2 I - TG'XG)^{-1}G'X)v \leq 0.$$

If

$$\left((2 - T\eta)\eta + 2\Re(\lambda) + T|\lambda|^2\right) v^*(X + TXG(\alpha^2 I - TG'XG)^{-1}G'X)v < 0,$$

then

$$v^*(X + TXG(\alpha^2 I - TG'XG)^{-1}G'X)v > 0$$

and

$$(2 - T\eta)\eta + 2\Re(\lambda) + T|\lambda|^2 < 0,$$

which reduces to the condition for $\lambda \in D_T^\eta$. If

$$\left((2 - T\eta)\eta + 2\Re(\lambda) + T|\lambda|^2\right) v^*(X + TXG(\alpha^2 I - TG'XG)^{-1}G'X)v = 0,$$

then all of the terms must be zero. In particular, $Hv = 0$. Since all unobservable modes of F have eigenvalues in D_T^η, $\lambda \in D_T^\eta$, which proves (a).

(b): Showing that

$$\|T_{wz}\|_\infty \leq \alpha$$

is equivalent to showing that

$$V := \|z\|_2{}^2 - \alpha^2\|w\|_2{}^2 \leq 0, \quad \forall w \in \mathcal{L}_2.$$

Note that

$$\mathcal{S}_{t=0}^{\infty} \frac{x(t+T)'Xx(t+T) - x(t)'Xx(t)}{T}\, dt = -x(0)'Xx(0) = 0.$$

We show that $V \leq 0$.

$$\begin{aligned}
V &= \|z\|_2{}^2 - \alpha^2\|w\|_2{}^2 - x(0)'Xx(0) \\
&= \mathcal{S}_{t=0}^{\infty}[z(t)'z(t) - \alpha^2 w(t)'w(t) \\
&\qquad + \tfrac{1}{T}(x(t+T)'Xx(t+T) - x(t)'Xx(t))]\, dt \\
&= \mathcal{S}_{t=0}^{\infty}[x(t)'\left(H'H + \tfrac{1}{T}(I+TF)'X(I+TF) - \tfrac{1}{T}X\right)x(t) \\
&\qquad + 2x(t)'\left((I+TF)'XG\right)w(t) + w(t)'\left(TG'XG - \alpha^2 I\right)w(t)]\, dt \\
&= \mathcal{S}_{t=0}^{\infty}[(x(t)'\left(H'H + F'X + XF + TF'XF\right))x(t) \\
&\qquad + 2x(t)'\left((I+TF)'XG\right)w(t) + w(t)'\left(TG'XG - \alpha^2 I\right)w(t)]\, dt.
\end{aligned}$$

But, by (i),

$$\begin{aligned}
&F'XL^{-1} + XL^{-1}F + TF'XL^{-1}F + \tfrac{1}{\alpha^2}XGG'XL^{-1} + H'H \\
&= H'H + F'X + XF + TF'XF \\
&\quad + (I+TF)'XG(\alpha^2 I - TG'XG)^{-1}G'X(I+TF) \\
&\leq -(2 - T\eta)\eta X \leq 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
V &\leq \mathcal{S}_{t=0}^{\infty}[x(t)'\left(-(I+TF)'XG(\alpha^2 I - TG'XG)^{-1}G'X(I+TF)\right)x(t) \\
&\qquad + 2x(t)'\left((I+TF)'XG\right)w(t) + w(t)'\left(-(\alpha^2 I - TG'XG)\right)w(t)]\, dt \\
&= -\mathcal{S}_{t=0}^{\infty}\left[w(t) - (\alpha^2 I - TG'XG)^{-1}G'X(I+TF)x(t)\right]' \\
&\qquad \cdot (\alpha^2 I - TG'XG)\left[w(t) - (\alpha^2 I - TG'XG)^{-1}G'X(I+TF)x(t)\right]\, dt \\
&\leq 0, \qquad \text{by (ii)}.
\end{aligned}$$

□

A.2 Derivations of the H_∞-norm-bounding Designs for Centralized Systems

First, the derivation is given of the unified formulation of the state-feedback controller in Theorem 2.3.1.

Theorem For the centralized system (2.1), (2.3), with (A, H) detectable and with state-feedback (2.4), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_0z}\|_\infty \leq \alpha$ is

$$K^c = -B'X\Lambda^{-1}(I + TA)$$

where

$$\Lambda := I + T(BB' - \frac{1}{\alpha^2}GG')X$$

and $X \geq 0$ satisfies

(i) $A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(BB' - \frac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H = 0$

(ii) $\alpha^2 I - TG'XG > 0.$

Proof Given the centralized system (2.1), (2.3), and the state-feedback (2.4), the resulting closed-loop system is

$$\rho x = (A + BK^c)x + Gw_0$$
$$z = \begin{pmatrix} H \\ K^c \end{pmatrix} x.$$

By the bounded real lemma, the sufficient condition is

$$\begin{aligned}(A + BK^c)'XL^{-1} + XL^{-1}(A + BK^c) + T(A + BK^c)'XL^{-1}(A + BK^c)\\ + \tfrac{1}{\alpha^2}XGG'XL^{-1} + (H' \; K^{c\prime})\begin{pmatrix} H \\ K^c \end{pmatrix} = 0\end{aligned} \tag{A.1}$$

and (ii) above. After some manipulations, using the matrix inversion lemma and cancelling terms, it can be shown that the first condition can be re-expressed as

$$\begin{aligned}0 = A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(BB' - \tfrac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H\\ + (K^c + B'X\Lambda^{-1}(I + TA))'(I + TB'XL^{-1}B)(K^c + B'X\Lambda^{-1}(I + TA)).\end{aligned}$$

Setting $K^c = -B'X\Lambda^{-1}(I + TA)$ results in the conditions in the theorem. □

Next, the derivation is presented for the centralized observer-based controller of Theorem 2.3.2.

Theorem For the centralized system (2.1), (2.2), (2.3), with (A, H) detectable and with observer-based feedback (2.5), (2.6), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_e z}\|_\infty \leq \alpha$ is

$$K^c = -B'X\Lambda^{-1}(I + TA), \qquad K^d = \frac{1}{\alpha^2}G'X\Lambda^{-1}(I + TA),$$

$$K^o = (I - \frac{1}{\alpha^2}YX)^{-1}(I + TA)\Pi^{-1}YC',$$

where

$$\Lambda := I + T(BB' - \frac{1}{\alpha^2}GG')X, \qquad \Pi := I + TY(C'C - \frac{1}{\alpha^2}H'H),$$

and $X \geq 0$ and $Y > 0$ satisfy

$$A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(BB' - \frac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H = 0$$

$$A\Pi^{-1}Y + \Pi^{-1}YA' + TA\Pi^{-1}YA' - \Pi^{-1}Y(C'C - \frac{1}{\alpha^2}H'H)Y + GG' = 0$$

such that

$$\rho(YX) < \alpha^2$$

and the eigenvalues of $A + GK^d - K^oC$ lie in D_T, and

$$\Xi(X,Y) := \begin{pmatrix} \alpha^2 I - TG'\alpha^2 Y^{-1}G & TG'(\alpha^2 Y^{-1} - X)K^o \\ TK^{o\prime}(\alpha^2 Y^{-1} - X)G & \alpha^2 I - TK^{o\prime}(\alpha^2 Y^{-1} - X)K^o \end{pmatrix} > 0$$

(or

$$X > 0, \qquad \alpha^2 I - TG'XG > 0,$$

$$|L^{-1}(I + TA) - TK^oC| \neq 0,$$

and

$$\alpha^2 Y^{-1} > (I + TA)'XL^{-1}(I + TA) + TH'H\,).$$

Proof Let $e := \xi - x$. Then the closed-loop system resulting from applying the observer-based feedback (2.5), (2.6), to the centralized system (2.1), (2.2), (2.3), is

$$\rho\begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} A+BK^c & BK^c \\ GK^d & A+GK^d-K^oC \end{pmatrix}\begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} G & 0 \\ -G & K^o \end{pmatrix}\begin{pmatrix} w_0 \\ w \end{pmatrix}$$

$$=: F_e\begin{pmatrix} x \\ e \end{pmatrix} + G_e w_e$$

$$z = \begin{pmatrix} H & 0 \\ K^c & K^c \end{pmatrix}\begin{pmatrix} x \\ e \end{pmatrix} =: H_e\begin{pmatrix} x \\ e \end{pmatrix}.$$

Applying the bounded real lemma, we obtain that the eigenvalues of F_e lie in D_T and $\|T_{w_e z}\|_\infty \le \alpha$ if (F_e, H_e) is detectable and there exist a real symmetric matrix $X_e \ge 0$ and a real $\alpha > 0$ such that

(i) $F_e' X_e L_e^{-1} + X_e L_e^{-1} F_e + T F_e' X_e L_e^{-1} F_e + \frac{1}{\alpha^2} X_e G_e G_e' X_e L_e^{-1} + H_e' H_e = 0$

(ii) $\alpha^2 I - T G_e' X_e G_e > 0$,

where $L_e := I - T\frac{1}{\alpha^2} G_e G_e' X_e$.

Suppose that X_e is of the form $\left(\begin{smallmatrix} X & 0 \\ 0 & X_1 \end{smallmatrix}\right)$. This form creates a separation between the resulting state and output Riccati equations, which is desirable for computability whenever it can be achieved.

Condition (i) can be manipulated to find conditions in terms of the basic matrices. First, let

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} := L_e^{-1}$$

$$= \begin{pmatrix} L & T\frac{1}{\alpha^2}GG'X_1 \\ T\frac{1}{\alpha^2}GG'X & I - T\frac{1}{\alpha^2}(GG' + K^oK^{o\prime})X_1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} L^{-1} + L^{-1}T\frac{1}{\alpha^2}GG'X_1D^{-1}T\frac{1}{\alpha^2}GG'XL^{-1} & -L^{-1}T\frac{1}{\alpha^2}GG'X_1D^{-1} \\ -D^{-1}T\frac{1}{\alpha^2}GG'XL^{-1} & D^{-1} \end{pmatrix}$$

where

$$D := I - T\frac{1}{\alpha^2}K^oK^{o\prime}X_1 - L^{-1}T\frac{1}{\alpha^2}GG'X_1.$$

The upper-left element in Equation (i) is then

$$\begin{aligned}
0 \;=\;& (A+BK^c)'X\Omega_{11} + (GK^d)'X_1\Omega_{21} + X\Omega_{11}(A+BK^c) + X\Omega_{12}(GK^d) \\
&+ H'H + K^{c\prime}K^c + \tfrac{1}{\alpha^2}XGG'X\Omega_{11} - \tfrac{1}{\alpha^2}XGG'X_1\Omega_{21} \\
&+ T(A+BK^c)'X\Omega_{11}(A+BK^c) + T(A+BK^c)'X\Omega_{12}(GK^d) \\
&+ T(GK^d)'X_1\Omega_{21}(A+BK^c) + T(GK^d)'X_1\Omega_{22}(GK^d) \\
\\
=\;& (A+BK^c)'XL^{-1} + XL^{-1}(A+BK^c) + T(A+BK^c)'XL^{-1}(A+BK^c) \\
&+ \tfrac{1}{\alpha^2}XGG'XL^{-1} + (H'\ K^{c\prime})\left(\begin{smallmatrix}H \\ K^c\end{smallmatrix}\right) \\
&+ T(\tfrac{1}{\alpha^2}GG'XL^{-1}(I+T(A+BK^c)) - GK^d)' \\
&\quad \cdot X_1D^{-1}(\tfrac{1}{\alpha^2}GG'XL^{-1}(I+T(A+BK^c)) - GK^d).
\end{aligned}$$

The first part of this equation is (A.1). If the last term is equal to 0, then the state Riccati Equation (2.14) results. The last term can be forced to be 0 by letting

$$GK^d = \frac{1}{\alpha^2}GG'XL^{-1}(I+TA+TBK^c),$$

which is satisfied if

$$K^d = \frac{1}{\alpha^2}G'XL^{-1}(I+TA+TBK^c).$$

Note that if

$$K^c = -B'X\Lambda^{-1}(I+TA),$$

then

$$K^d = \frac{1}{\alpha^2}G'XL^{-1}(\Lambda - TBB'X)\Lambda^{-1}(I+TA) = \frac{1}{\alpha^2}G'X\Lambda^{-1}(I+TA).$$

From the upper-right element of (i), we find, after cancelling terms, that

$$0 = (I+T(A+BK^c))'XL^{-1}(BK^c) + K^{c\prime}K^c,$$

which is satisfied if

$$K^c = -B'XL^{-1}(I+TA+TBK^c).$$

But $K^c = -B'X\Lambda^{-1}(I+TA)$ satisfies this. Thus, $K^d = \frac{1}{\alpha^2}G'X\Lambda^{-1}(I+TA)$ satisfies the upper-left element's equation. The lower-left element of (i) reduces to the upper-right element of (i). The lower-right element of (i) can be rearranged to

$$\begin{aligned}
0 = {} & T(A + GK^d - K^oC - T\tfrac{1}{\alpha^2}GG'XL^{-1}BK^c)'X_1D^{-1} \\
& \cdot (A + GK^d - K^oC - T\tfrac{1}{\alpha^2}GG'XL^{-1}BK^c) + T(BK^c)'XL^{-1}(BK^c) \\
& + K^{c\prime}K^c + (A + GK^d - K^oC - T\tfrac{1}{\alpha^2}GG'XL^{-1}BK^c)'X_1D^{-1} \\
& + X_1D^{-1}(A + GK^d - K^oC - T\tfrac{1}{\alpha^2}GG'XL^{-1}BK^c) \\
& + (\tfrac{1}{\alpha^2}X_1GG'XL^{-1}T\tfrac{1}{\alpha^2}GG' + \tfrac{1}{\alpha^2}X_1GG' + \tfrac{1}{\alpha^2}X_1K^oK^{o\prime})X_1D^{-1}.
\end{aligned}$$

This reduces the problem to two cases—the continuous-time case if $T = 0$ and the discrete-time case if $T \neq 0$.

First consider the case $T = 0$. Note that $X_1D^{-1} \to X_1$ as $T \to 0$. The equations then reduce to

$$\begin{aligned}
0 = {} & K^{c\prime}K^c + (A + GK^d - K^oC)'X_1 + X_1(A + GK^d - K^oC) \\
& + \tfrac{1}{\alpha^2}X_1GG'X_1 + \tfrac{1}{\alpha^2}X_1K^oK^{o\prime}X_1.
\end{aligned}$$

Here, $K^c = -B'X$ and $K^d = \frac{1}{\alpha^2}G'X$, and thus

$$\begin{aligned}
0 = {} & XBB'X + (A + \tfrac{1}{\alpha^2}GG'X - K^oC)'X_1 + X_1(A + \tfrac{1}{\alpha^2}GG'X - K^oC) \\
& + \tfrac{1}{\alpha^2}X_1GG'X_1 + \tfrac{1}{\alpha^2}X_1K^oK^{o\prime}X_1.
\end{aligned}$$

From [25], this is satisfied by the design equations

$$K^o = (I - \frac{1}{\alpha^2}YX)^{-1}YC', \qquad Y := \alpha^2(X + X_1)^{-1}, \qquad \rho(YX) < \alpha^2$$

$$AY + YA' + \frac{1}{\alpha^2}YH'HY - YC'CY + GG' = 0,$$

which are also the limit of Equations (2.12), (2.16), and (2.15) as $T \to 0$.

Now consider the case $T \neq 0$. Multiplying both sides of the equation by T and adding and subtracting X_1D^{-1}, we obtain

$$\begin{aligned}
X_1 \ = \ & (I + T(A + GK^d - K^oC - T\tfrac{1}{\alpha^2}GG'XL^{-1}BK^c))'X_1D^{-1} \\
& \cdot (I + T(A + GK^d - K^oC - T\tfrac{1}{\alpha^2}GG'XL^{-1}BK^c)) \\
& + TK^{c\prime}K^c + (TBK^c)'XL^{-1}(TBK^c).
\end{aligned}$$

This simplifies to

$$\begin{aligned}
X_1 = {} & (L^{-1}(I + TA) - TK^oC)'X_1D^{-1}(L^{-1}(I + TA) - TK^oC) \\
& + (I + TA)'XL^{-1}(I + TA) - (I + TA)'X\Lambda^{-1}(I + TA).
\end{aligned} \tag{A.2}$$

Multiplying both sides of Equation (2.14) by T, we find that

$$X = (I + TA)'X\Lambda^{-1}(I + TA) + TH'H,$$

which we add to the previous equation to obtain, after simplification,

$$
\begin{aligned}
X + X_1 &= (L^{-1}(I+TA) - TK^{\circ}C)'X_1D^{-1}(L^{-1}(I+TA) - TK^{\circ}C) \\
&\quad + (I+TA)'XL^{-1}(I+TA) + TH'H.
\end{aligned}
$$

Let $Y := \alpha^2(X + X_1)^{-1}$. We require that $X_1 > 0$ and $X \geq 0$, so that $(X + X_1)$ is invertible and $X_e \geq 0$. These assumptions on X and X_1 are equivalent to requiring that $Y > 0$ and $\rho(YX) < \alpha^2$ since $X_1{}^{-1} = \frac{1}{\alpha^2}(I - \frac{1}{\alpha^2}YX)^{-1}Y$ and $X + X_1 = \alpha^2 Y^{-1}$ and Y is a real, symmetric matrix. Substituting, we find that

$$
\begin{aligned}
\alpha^2 Y^{-1} &= (I+TA)'XL^{-1}(I+TA) + TH'H \\
&\quad + (L^{-1}(I+TA) - TK^{\circ}C)' \\
&\quad\quad \cdot (\tfrac{1}{\alpha^2}(I - \tfrac{1}{\alpha^2}YX)^{-1}Y - T\tfrac{1}{\alpha^2}K^{\circ}K^{\circ\prime} - L^{-1}T\tfrac{1}{\alpha^2}GG')^{-1} \\
&\quad\quad \cdot (L^{-1}(I+TA) - TK^{\circ}C).
\end{aligned}
$$

Multiplying on the left by $(I+TA)\Pi^{-1}Y$ and on the right by $\Pi^{-1}Y(I+TA)'$, where Π is defined by (2.13), and simplifying,

$$
\begin{aligned}
&\alpha^2(I+TA)\Pi^{-1}Y(I+TA)' \\
&\quad = (I+TA)\Pi^{-1}Y(I+TA)'XL^{-1}(I+TA)\Pi^{-1}Y(I+TA)' \\
&\quad\quad + \alpha^2(I+TA)\Pi^{-1}YTC'C\Pi^{-1}Y(I+TA)' \\
&\quad\quad + (I+TA)\Pi^{-1}Y(L^{-1}(I+TA) - TK^{\circ}C)'(I - \tfrac{1}{\alpha^2}XY)\Phi^{-1} \\
&\quad\quad\quad \cdot (I - \tfrac{1}{\alpha^2}YX)(L^{-1}(I+TA) - TK^{\circ}C)\Pi^{-1}Y(I+TA)'
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi &:= (I - \tfrac{1}{\alpha^2}YX)(\tfrac{1}{\alpha^2}(I - \tfrac{1}{\alpha^2}YX)^{-1}Y - L^{-1}T\tfrac{1}{\alpha^2}GG')(I - \tfrac{1}{\alpha^2}XY) \\
&\quad - (I - \tfrac{1}{\alpha^2}YX)T\tfrac{1}{\alpha^2}K^{\circ}K^{\circ\prime}(I - \tfrac{1}{\alpha^2}XY) \\
&= \tfrac{1}{\alpha^2}(I - \tfrac{1}{\alpha^2}YX)(I - T\tfrac{1}{\alpha^2}GG'X)^{-1}(Y - TGG') \\
&\quad - (I - \tfrac{1}{\alpha^2}YX)T\tfrac{1}{\alpha^2}K^{\circ}K^{\circ\prime}(I - \tfrac{1}{\alpha^2}XY).
\end{aligned}
$$

Thus,

$$\alpha^2(I+TA)\Pi^{-1}Y(I+TA)'$$

$$\begin{aligned}
&= (I+TA)\Pi^{-1}Y(I+TA)'XL^{-1}(I+TA)\Pi^{-1}Y(I+TA)' \\
&\quad + \alpha^2(I+TA)\Pi^{-1}YTC'C\Pi^{-1}Y(I+TA)' \\
&\quad + \alpha^2[(I+TA)\Pi^{-1}Y(I+TA)'L^{-1\prime}(I-\tfrac{1}{\alpha^2}XY) \\
&\qquad - (I+TA)\Pi^{-1}YC'TK^{o\prime}(I-\tfrac{1}{\alpha^2}XY)] \\
&\quad \cdot [(I-\tfrac{1}{\alpha^2}YX)L^{-1}(Y-TGG') - (I-\tfrac{1}{\alpha^2}YX)TK^oK^{o\prime}(I-\tfrac{1}{\alpha^2}XY)]^{-1} \\
&\quad \cdot [(I-\tfrac{1}{\alpha^2}YX)L^{-1}(I+TA)\Pi^{-1}Y(I+TA)' \\
&\qquad - (I-\tfrac{1}{\alpha^2}YX)TK^oC\Pi^{-1}Y(I+TA)'].
\end{aligned}$$

Noting that

$$\alpha^2 I - XL^{-1}(I+TA)\Pi^{-1}Y(I+TA)' + \alpha^2 L^{-1\prime} - L^{-1\prime}XY$$

$$= XL^{-1}(Y - TGG' - (I+TA)\Pi^{-1}Y(I+TA)'),$$

we see that

$$Y - TGG' = (I+TA)\Pi^{-1}Y(I+TA)' \tag{A.3}$$

and

$$K^{o\prime}(I - \frac{1}{\alpha^2}XY) = C\Pi^{-1}Y(I+TA)' \tag{A.4}$$

satisfy this condition. The first of these, (A.3), can be reduced to (2.15) by recalling that $T \neq 0$ and dividing both sides by T, and the second, (A.4), is equivalent to (2.12).

Now we still must check the detectability of (F_e, H_e) and find conditions for (ii) to hold.

First, we check the detectability condition. We assume that (A, H) is detectable and that the eigenvalues of $A + GK^d - K^oC$ lie in D_T and must show that it follows that (F_e, H_e) is detectable. Let $v = \binom{v_1}{v_2} \neq 0$ satisfy

$$F_e v = \begin{pmatrix} A + BK^c & BK^c \\ GK^d & A + GK^d - K^oC \end{pmatrix} v = \lambda v \tag{A.5}$$

$$H_e v = \begin{pmatrix} H & 0 \\ K^c & K^c \end{pmatrix} v = 0. \tag{A.6}$$

From (A.6), $Hv_1 = 0$ and $K^c(v_1 + v_2) = 0$. From (A.5),

$$\lambda v_1 = Av_1 + BK^c(v_1 + v_2) = Av_1.$$

Since (A, H) is detectable, λ is in the stability region or $v_1 = 0$. Suppose that $v_1 = 0$. Then, $(A + GK^d - K^oC)v_2 = \lambda v_2$, and thus $\lambda \in D_T$. Hence, (F_e, H_e) is detectable.

Condition (ii) is

$$\alpha^2 I - TG_e' X_e G_e > 0, \tag{A.7}$$

which can be expanded as (2.17). If we assume further that $X > 0$, then (A.7) is equivalent to

$$\begin{aligned}
0 &< X_e^{-1} - T\tfrac{1}{\alpha^2}G_eG_e' \\
&= \begin{pmatrix} X^{-1} - T\frac{1}{\alpha^2}GG' & T\frac{1}{\alpha^2}GG' \\ T\frac{1}{\alpha^2}GG' & X^{-1} - T\frac{1}{\alpha^2}(GG' + K^oK^{o\prime}) \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ T\frac{1}{\alpha^2}GG'XL^{-1} & I \end{pmatrix} \begin{pmatrix} LX^{-1} & 0 \\ 0 & DX_1^{-1} \end{pmatrix} \begin{pmatrix} I & T\frac{1}{\alpha^2}XL^{-1}GG' \\ 0 & I \end{pmatrix},
\end{aligned}$$

or to

$$\alpha^2 I - TG'XG > 0$$

and, by (A.2), as long as $|L^{-1}(I + TA) - TK^oC| \neq 0$, which is reasonable for small T,

$$\begin{aligned}
0 &< X_1 - (I + TA)'XL^{-1}(I + TA) + (I + TA)'X\Lambda^{-1}(I + TA) \\
&= X_1 + X - TH'H - (I + TA)'XL^{-1}(I + TA),
\end{aligned}$$

or, equivalently,

$$\alpha^2 Y^{-1} > (I + TA)'XL^{-1}(I + TA) + TH'H.$$

□

A.3 Derivations of the Reliable Designs for Centralized Systems

First, the derivation of the sensor-outage reliable controller design from Theorem 2.3.3 is presented.

Theorem For the centralized system (2.1), (2.2), (2.3), with (A, H) detectable and with output feedback (2.5), (2.6), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_e z}\|_\infty \leq \alpha$ for all sensor failures $\omega \subseteq \Omega$ is

$$K^c = -B'X\Lambda^{-1}(I + TA), \qquad K^d = \frac{1}{\alpha^2}G'X\Lambda^{-1}(I + TA),$$

$$K^o = (I - \frac{1}{\alpha^2}YX)^{-1}(I + TA)\Pi^{-1}YC'$$

where

$$\Lambda := I + T(BB' - \frac{1}{\alpha^2}GG')X, \qquad \Pi := I + TY(C_{\bar{\Omega}}'C_{\bar{\Omega}} - \frac{1}{\alpha^2}H'H)$$

and $X \geq 0$ and $Y > 0$ satisfy

$$A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(BB' - \frac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H + \alpha^2 C_\Omega{}'C_\Omega = 0$$

$$A\Pi^{-1}Y + \Pi^{-1}YA' + TA\Pi^{-1}YA' - \Pi^{-1}Y(C_{\bar{\Omega}}'C_{\bar{\Omega}} - \frac{1}{\alpha^2}H'H)Y + GG' = 0$$

such that

$$\rho(YX) < \alpha^2$$

and the eigenvalues of $A + GK^d - K^oC$ lie in D_T, and

$$\Xi(X,Y) > 0$$

and, for all $\omega \subseteq \Omega$,

$$\bar{\Xi}(X,Y) := \begin{pmatrix} \alpha^2 I - TG'\alpha^2 Y^{-1}G & TG'(\alpha^2Y^{-1} - X)K^o_\omega \\ TK^o_\omega{}'(\alpha^2Y^{-1} - X)G & \alpha^2 I - TK^o_\omega{}'(\alpha^2Y^{-1} - X)K^o_\omega \end{pmatrix} > 0$$

(or, instead of the conditions on $\Xi(X,Y)$ and $\bar{\Xi}(X,Y)$,

$$X > 0, \qquad \alpha^2 I - TG'XG > 0,$$

$$|L^{-1}(I + TA) - TK^oC| \neq 0,$$

and

$$\alpha^2 Y^{-1} > (I + TA)'XL^{-1}(I + TA) + TH'H + T\alpha^2 C_\Omega{}'C_\Omega\,).$$

Proof Consider the closed-loop plant with and without sensor failures $\omega \subseteq \Omega$. The system without outages is

$$F_e = \begin{pmatrix} A + BK^c & BK^c \\ GK^d & A + GK^d - K^oC \end{pmatrix}, \qquad G_e = \begin{pmatrix} G & 0 \\ -G & K^o \end{pmatrix},$$

$$H_e = \begin{pmatrix} H & 0 \\ K^c & K^c \end{pmatrix}.$$

The system with outages is

$$\bar{F}_e = \begin{pmatrix} A + BK^c & BK^c \\ GK^d - K^o_\omega C_\omega & A + GK^d - K^oC \end{pmatrix} = F_e - \begin{pmatrix} 0 & 0 \\ K^o_\omega C_\omega & 0 \end{pmatrix},$$

$$\bar{G}_e = \begin{pmatrix} G & 0 \\ -G & K^o_\omega \end{pmatrix} = G_e - \begin{pmatrix} 0 & 0 \\ 0 & K^o_\omega \end{pmatrix},$$

$$\bar{H}_e = H_e.$$

Thus,

$$\bar{F}_e = F_e - K^o_{\omega e} C_{\omega e}$$

$$\bar{G}_e \bar{G}_e' = G_e G_e' - K^o_{\omega e} K^o_{\omega e}{}'$$

$$\bar{H}_e' \bar{H}_e = H_e' H_e$$

where

$$K^o_{\omega e} := \begin{pmatrix} 0 \\ K^o_\omega \end{pmatrix}, \qquad C_{\omega e} := (C_\omega \ 0).$$

We would like to see how the Riccati equations need to be changed to satisfy the bounded real lemma when sensor outages occur. Consider condition (i) of the bounded real lemma. First, note that

$$\begin{aligned} \bar{L}_e^{-1} &= (I - T\tfrac{1}{\alpha^2}\bar{G}_e\bar{G}_e' X_e)^{-1} \\ &= (L_e - T\tfrac{1}{\alpha^2}K^o_{\omega e}K^{o\prime}_{\omega e}X_e)^{-1} \\ &= L_e{}^{-1} - L_e{}^{-1}T\tfrac{1}{\alpha^2}K^o_{\omega e}(I + T\tfrac{1}{\alpha^2}K^{o\prime}_{\omega e}X_e L_e{}^{-1}K^o_{\omega e})^{-1}K^{o\prime}_{\omega e}X_e L_e{}^{-1}. \end{aligned}$$

Therefore, condition (i) becomes

$$\bar{F}_e' X_e \bar{L}_e^{-1} + X_e \bar{L}_e^{-1}\bar{F}_e + T\bar{F}_e' X_e \bar{L}_e^{-1}\bar{F}_e + \tfrac{1}{\alpha^2}X_e\bar{G}_e\bar{G}_e' X_e \bar{L}_e^{-1} + \bar{H}_e'\bar{H}_e$$

$$\begin{aligned} &= F_e{}' X_e L_e{}^{-1} + X_e L_e{}^{-1} F_e + T F_e{}' X_e L_e{}^{-1} F_e + \tfrac{1}{\alpha^2} X_e G_e G_e{}' X_e L_e{}^{-1} \\ &\quad + H_e{}' H_e + \alpha^2 C_{\omega e}' C_{\omega e} - (\alpha C_{\omega e} + \tfrac{1}{\alpha}K^{o\prime}_{\omega e} X_e L_e{}^{-1}(I + TF_e))' \\ &\quad \cdot (I + T\tfrac{1}{\alpha^2}K^{o\prime}_{\omega e}X_e L_e{}^{-1}K^o_{\omega e})^{-1}(\alpha C_{\omega e} + \tfrac{1}{\alpha}K^{o\prime}_{\omega e}X_e L_e{}^{-1}(I + TF_e)). \end{aligned}$$

Let $X_e \geq 0$ satisfy

$$\begin{aligned} &F_e{}' X_e L_e{}^{-1} + X_e L_e{}^{-1} F_e + T F_e{}' X_e L_e{}^{-1} F_e + \tfrac{1}{\alpha^2} X_e G_e G_e' X_e L_e{}^{-1} + H_e' H_e \\ &\quad + \alpha^2 C_{\omega e}' C_{\omega e} = 0. \end{aligned}$$

Then,

$$\bar{F}_e'X_e\bar{L}_e^{-1} + X_e\bar{L}_e^{-1}\bar{F}_e + T\bar{F}_e'X_e\bar{L}_e^{-1}\bar{F}_e + \tfrac{1}{\alpha^2}X_e\bar{G}_e\bar{G}_e'X_e\bar{L}_e^{-1} + \bar{H}_e'\bar{H}_e$$

$$\begin{aligned} &= -(\alpha C_{\omega e} + \tfrac{1}{\alpha}K_{\omega e}^{o\prime}X_eL_e^{-1}(I+TF_e))'(I + T\tfrac{1}{\alpha^2}K_{\omega e}^{o\prime}X_eL_e^{-1}K_{\omega e}^{o})^{-1} \\ &\quad \cdot(\alpha C_{\omega e} + \tfrac{1}{\alpha}K_{\omega e}^{o\prime}X_eL_e^{-1}(I+TF_e)) \\ &\le 0 \end{aligned}$$

if

$$\begin{aligned} 0 &< I + T\tfrac{1}{\alpha^2}K_{\omega e}^{o\prime}X_eL_e^{-1}K_{\omega e}^{o} \\ &= I + T\tfrac{1}{\alpha^2}K_{\omega e}^{o\prime}(X_e + TX_eG_e(\alpha^2 I - TG_e'X_eG_e)^{-1}G_e'X_e)K_{\omega e}^{o}. \end{aligned}$$

This condition is satisfied if $\alpha^2 I - TG_e'X_eG_e > 0$.

But the Riccati equation for X_e depended on ω and we want this design to work for all $\omega \subseteq \Omega$. Let $X_e \ge 0$ be such that

$$\begin{aligned} &F_e'X_eL_e^{-1} + X_eL_e^{-1}F_e + TF_e'X_eL_e^{-1}F_e + \tfrac{1}{\alpha^2}X_eG_eG_e'X_eL_e^{-1} + H_e'H_e \\ &+ \alpha^2 C_{\Omega e}'C_{\Omega e} = 0 \end{aligned} \tag{A.8}$$

where

$$C_{\Omega e} := (C_\Omega \ 0).$$

Then,

$$\bar{F}_e'X_e\bar{L}_e^{-1} + X_e\bar{L}_e^{-1}\bar{F}_e + T\bar{F}_e'X_e\bar{L}_e^{-1}\bar{F}_e + \tfrac{1}{\alpha^2}X_e\bar{G}_e\bar{G}_e'X_e\bar{L}_e^{-1} + \bar{H}_e'\bar{H}_e$$

$$\begin{aligned} &= -(\alpha C_{\omega e} + \tfrac{1}{\alpha}K_{\omega e}^{o\prime}X_eL_e^{-1}(I+TF_e))'(I + T\tfrac{1}{\alpha^2}K_{\omega e}^{o\prime}X_eL_e^{-1}K_{\omega e}^{o})^{-1} \\ &\quad \cdot(\alpha C_{\omega e} + \tfrac{1}{\alpha}K_{\omega e}^{o\prime}X_eL_e^{-1}(I+TF_e)) \\ &\quad + \alpha^2(C_{\omega e}'C_{\omega e} - C_{\Omega e}'C_{\Omega e}) \\ &\le 0 \end{aligned}$$

if $\Xi(X,Y) = \alpha^2 I - TG_e'X_eG_e > 0$.

Next, we must find out how the additional term in (A.8) affects the individual Riccati equations. The constant term (not containing X_e) in (A.8) is

$$H_e'H_e + \alpha^2 C_{\Omega e}'C_{\Omega e} = \begin{pmatrix} (H'H + \alpha^2 C_\Omega{}'C_\Omega) + K^{c\prime}K^c & K^{c\prime}K^c \\ K^{c\prime}K^c & K^{c\prime}K^c \end{pmatrix}.$$

Also, H does not appear in any other term of (A.8), and thus (A.8) is equivalent to condition (i) of the bounded real lemma applied to the standard case with every $H'H$ replaced with $H'H+\alpha^2 C_\Omega{}'C_\Omega$. Thus, the conditions can be found by replacing $H'H$ with $H'H + \alpha^2 C_\Omega{}'C_\Omega$. Hence,

$$\begin{aligned} \Pi &:= I + TY(C'C - \tfrac{1}{\alpha^2}(H'H + \alpha^2 C_\Omega{}'C_\Omega)) \\ &= I + TY(C_{\bar\Omega}{}'C_{\bar\Omega} - \tfrac{1}{\alpha^2}H'H) \end{aligned}$$

and $X \geq 0$ and $Y > 0$ satisfy (2.18) and (2.19). The gain K^o is changed only through the changes in X, Y, and Π.

Next, we must check how condition (ii) of the bounded real lemma is affected. Condition (ii) becomes, for all $\omega \subseteq \Omega$,

$$\begin{aligned} 0 &< \alpha^2 I - T\bar{G}_e'X_e\bar{G}_e \\ &= \alpha^2 I - T\begin{pmatrix} G & 0 \\ -G & K_\omega^o \end{pmatrix}'\begin{pmatrix} X & 0 \\ 0 & X_1 \end{pmatrix}\begin{pmatrix} G & 0 \\ -G & K_\omega^o \end{pmatrix} \\ &= \begin{pmatrix} \alpha^2 I - TG'\alpha^2 Y^{-1}G & TG'(\alpha^2Y^{-1} - X)K_\omega^o \\ TK_\omega^{o\prime}(\alpha^2 Y^{-1} - X)G & \alpha^2 I - TK_\omega^{o\prime}(\alpha^2Y^{-1} - X)K_\omega^o \end{pmatrix} \\ &=: \bar\Xi(X,Y). \end{aligned}$$

Alternatively, if we assume that $X > 0$, then

$$0 < \alpha^2 I - T\bar{G}_e'X_e\bar{G}_e$$

is equivalent to

$$\begin{aligned} 0 &< X_e{}^{-1} - T\tfrac{1}{\alpha^2}\bar{G}_e\bar{G}_e' \\ &= \begin{pmatrix} X^{-1} - T\frac{1}{\alpha^2}GG' & T\frac{1}{\alpha^2}GG' \\ T\frac{1}{\alpha^2}GG' & X_1{}^{-1} - T\frac{1}{\alpha^2}(GG' + K^oK^{o\prime} - K_\omega^oK_\omega^{o\prime}) \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ T\frac{1}{\alpha^2}GG'XL^{-1} & I \end{pmatrix}\begin{pmatrix} LX^{-1} & 0 \\ 0 & DX_1{}^{-1} + T\frac{1}{\alpha^2}K_\omega^oK_\omega^{o\prime} \end{pmatrix}\begin{pmatrix} I & T\frac{1}{\alpha^2}XL^{-1}GG' \\ 0 & I \end{pmatrix}. \end{aligned}$$

This is equivalent to

$$\alpha^2 I - TG'XG > 0$$

and

$$0 < DX_1{}^{-1} + T\frac{1}{\alpha^2}K^o_\omega K^{o\prime}_\omega. \tag{A.9}$$

But (A.9) holds if $0 < DX_1{}^{-1}$. This is also the condition that forces $\Xi(X,Y) > 0$. Therefore, conditions for this will satisfy both $\Xi(X,Y) > 0$ and $\bar{\Xi}(X,Y) > 0$. By (A.2), $0 < DX_1{}^{-1}$ is satisfied, as long as $|L^{-1}(I+TA) - TK^oC| \neq 0$, if

$$\begin{aligned} 0 &< X_1 - (I+TA)'XL^{-1}(I+TA) + (I+TA)'X\Lambda^{-1}(I+TA) \\ &= X_1 + X - TH'H - T\alpha^2 C_\Omega{}' C_\Omega - (I+TA)'XL^{-1}(I+TA), \end{aligned}$$

or, equivalently,

$$\alpha^2 Y^{-1} > (I+TA)'XL^{-1}(I+TA) + TH'H + T\alpha^2 C_\Omega{}' C_\Omega.$$

We also check the detectability condition. We must check if

$$(\bar{F}_e, \bar{H}_e) = (F_e - K^o_{\omega e}C_{\omega e}, H_e)$$

is detectable. Suppose that $v = \binom{v_1}{v_2} \neq 0$ satisfies

$$(F_e - K^o_{\omega e}C_{\omega e})v = \lambda v, \qquad H_e v = 0.$$

Then,

$$Av_1 = \lambda v, \qquad Hv_1 = 0.$$

Since (A, H) is assumed detectable, then either λ is in the stability region or $v_1 = 0$. If $v_1 = 0$, then

$$\bar{F}_e \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = F_e \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ v_2 \end{pmatrix}$$

and

$$H_e \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = 0.$$

This reduces to the standard case; then, since the eigenvalues of $A + GK^d - K^oC$ lie in D_T, λ is in the stability region. □

Next, the derivation is given for the actuator-outage reliable controller design in Theorem 2.3.4.

Theorem For the centralized system (2.1), (2.2), (2.3), with (A, H) detectable and with output feedback

$$u = K^c\xi, \quad \hat{w}_0 = K^d\xi, \quad \hat{u}_\Omega = K^\Omega\xi,$$

$$\rho\xi = A\xi + Bu + G\hat{w}_0 - B_\Omega\hat{u}_\Omega + K^o(y - C\xi),$$

a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_ez}\|_\infty \leq \alpha$ for all actuator failures $\omega \subseteq \Omega$ is

$$K^c = -B'X\Lambda^{-1}(I + TA), \qquad K^d = \frac{1}{\alpha^2}G'X\Lambda^{-1}(I + TA),$$

$$K^\Omega = -B_\Omega{}'X\Lambda^{-1}(I + TA),$$

$$K^o = (I - \frac{1}{\alpha^2}YX)^{-1}(I + TA)\Pi^{-1}YC'$$

where

$$\Lambda := I + T(BWbB_{\bar{\Omega}}{}' - \frac{1}{\alpha^2}GG')X, \qquad \Pi := I + TY(C'C - \frac{1}{\alpha^2}H'H)$$

and $X \geq 0$ and $Y > 0$ satisfy

$$A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(B_{\bar{\Omega}}B_{\bar{\Omega}}{}' - \frac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H = 0$$

$$A\Pi^{-1}Y + \Pi^{-1}YA' + TA\Pi^{-1}YA' - \Pi^{-1}Y(C'C - \frac{1}{\alpha^2}H'H)Y + GG' + \alpha^2 B_\Omega B_\Omega{}' = 0$$

such that

$$\rho(YX) < \alpha^2$$

and the eigenvalues of $A + GK^d - K^oC + BK^c$ lie in D_T and

$$I - T(B_\Omega{}' \;\; -B_\Omega{}')X_eL_e{}^{-1}\begin{pmatrix} B_\Omega \\ -B_\Omega{}' \end{pmatrix} > 0$$

(or

$$X > 0, \qquad X^{-1} - T\frac{1}{\alpha^2}GG' - TB_\Omega B_\Omega{}' > 0,$$

$$|(L - TB_\Omega B_\Omega{}'X)^{-1}(I + TA) - TK^oC| \neq 0,$$

and

$$\alpha^2Y^{-1} > (I + TA)'X(L - TB_\Omega B_\Omega{}'X)^{-1}(I + TA) + TH'H\,).$$

Proof If we set $u = K^c\xi$ in the design, then, when failures occur, the actual control becomes $u = K^c_{\bar{\omega}}\xi$. Thus, the actual controlled system will be

$$\rho x = Ax + BK^c\xi - B_\omega K^c_\omega\xi + Gw_0.$$

Assume the original form for the observer, with no $\hat{u}_\Omega$ term. The resulting system without actuator outages is

$$F_e = \begin{pmatrix} A + BK^c & BK^c \\ GK^d & A + GK^d - K^oC \end{pmatrix},$$

$$G_e = \begin{pmatrix} G & 0 \\ -G & K^o \end{pmatrix}, \qquad H_e = \begin{pmatrix} H & 0 \\ K^c & K^c \end{pmatrix}.$$

The system with actuator outages is

$$\begin{aligned} \bar{F}_e &= \begin{pmatrix} A + BK^c - B_\omega K_\omega^c & BK^c - B_\omega K_\omega^c \\ GK^d + B_\omega K_\omega^c & A + GK^d - K^oC + B_\omega K_\omega^c \end{pmatrix} \\ &= F_e - \begin{pmatrix} B_\omega K_\omega^c & B_\omega K_\omega^c \\ -B_\omega K_\omega^c & -B_\omega K_\omega^c \end{pmatrix}, \end{aligned}$$

$$\bar{G}_e = G_e, \qquad \bar{H}_e = \begin{pmatrix} H & 0 \\ K^c - K_\omega^c & K^c - K_\omega^c \end{pmatrix} = H_e - \begin{pmatrix} 0 & 0 \\ K_\omega^c & K_\omega^c \end{pmatrix}.$$

Therefore,

$$\bar{F}_e = F_e - B_{\omega e} K_{\omega e}^c$$

and

$$\bar{H}_e' \bar{H}_e = H_e' H_e - K_{\omega e}^{c\prime} K_{\omega e}^c.$$

where

$$B_{\omega e} := \begin{pmatrix} B_\omega \\ -B_\omega \end{pmatrix}, \qquad K_{\omega e}^c := (K_\omega^c \ K_\omega^c).$$

We would like to see how the Riccati equations need to be changed to satisfy the bounded real lemma when actuator outages occur. Consider condition (i) of the bounded real lemma. First, note that $\bar{L}_e^{-1} = L_e{}^{-1}$. Then,

$$\bar{F}_e' X_e \bar{L}_e^{-1} + X_e \bar{L}_e^{-1} \bar{F}_e + T \bar{F}_e' X_e \bar{L}_e^{-1} \bar{F}_e + \tfrac{1}{\alpha^2} X_e \bar{G}_e \bar{G}_e' X_e \bar{L}_e^{-1} + \bar{H}_e' \bar{H}_e$$

$$\begin{aligned}
&= F_e' X_e L_e{}^{-1} + X_e L_e{}^{-1} F_e + T F_e' X_e L_e{}^{-1} F_e + \tfrac{1}{\alpha^2} X_e G_e G_e' X_e L_e{}^{-1} + H_e' H_e \\
&\quad - K_{\omega e}^{c\prime} B_{\omega e}' X_e L_e{}^{-1}(I + TF_e) - (I + TF_e)' X_e L_e{}^{-1} B_{\omega e} K_{\omega e}^{c} \\
&\quad + T K_{\omega e}^{c\prime} B_{\omega e}' X_e L_e{}^{-1} B_{\omega e} K_{\omega e}^{c} - K_{\omega e}^{c\prime} K_{\omega e}^{c}
\end{aligned}$$

$$\begin{aligned}
&= F_e' X_e L_e{}^{-1} + X_e L_e{}^{-1} F_e + T F_e' X_e L_e{}^{-1} F_e + \tfrac{1}{\alpha^2} X_e G_e G_e' X_e L_e{}^{-1} + H_e' H_e \\
&\quad + (I + TF_e)' X_e L_e{}^{-1} B_{\Omega e}(I - T B_{\Omega e}' X_e L_e{}^{-1} B_{\Omega e})^{-1} B_{\Omega e}' X_e L_e{}^{-1}(I + TF_e) \\
&\quad - [K_{\omega e}^{c\prime}(I - T B_{\omega e}' X_e L_e{}^{-1} B_{\omega e}) + (I + TF_e)' X_e L_e{}^{-1} B_{\omega e}] \\
&\qquad \cdot (I - T B_{\omega e}' X_e L_e{}^{-1} B_{\omega e})^{-1} \\
&\qquad \cdot [(I - T B_{\omega e}' X_e L_e{}^{-1} B_{\omega e}) K_{\omega e}^{c} + B_{\omega e}' X_e L_e{}^{-1}(I + TF_e)] \\
&\quad + (I + TF_e)' X_e L_e{}^{-1} B_{\omega e}(I - T B_{\omega e}' X_e L_e{}^{-1} B_{\omega e})^{-1} B_{\omega e}' X_e L_e{}^{-1}(I + TF_e) \\
&\quad - (I + TF_e)' X_e L_e{}^{-1} B_{\Omega e}(I - T B_{\Omega e}' X_e L_e{}^{-1} B_{\Omega e})^{-1} B_{\Omega e}' X_e L_e{}^{-1}(I + TF_e).
\end{aligned}$$

If $X_e > 0$ and $I - T B_{\Omega e}' X_e L_e{}^{-1} B_{\Omega e} > 0$, then $(X_e L_e{}^{-1})^{-1} - T B_{\Omega e} B_{\Omega e}' > 0$ since

$$\begin{aligned}
&\left((X_e L_e{}^{-1})^{-1} - T B_{\Omega e} B_{\Omega e}'\right)^{-1} \\
&= X_e L_e{}^{-1} + X_e L_e{}^{-1} T B_{\Omega e}(I - T B_{\Omega e}' X_e L_e{}^{-1} B_{\Omega e})^{-1} B_{\Omega e}' X_e L_e{}^{-1}.
\end{aligned}$$

From this,

$$(X_e L_e{}^{-1})^{-1} - T B_{\omega e} B_{\omega e}' \geq (X_e L_e{}^{-1})^{-1} - T B_{\Omega e} B_{\Omega e}' > 0,$$

and thus

$$\left((X_e L_e{}^{-1})^{-1} - T B_{\omega e} B_{\omega e}'\right)^{-1} \leq \left((X_e L_e{}^{-1})^{-1} - T B_{\Omega e} B_{\Omega e}'\right)^{-1}.$$

Expanding both sides, we find that

$$B_{\omega e}(I - T B_{\omega e}' X_e L_e{}^{-1} B_{\omega e})^{-1} B_{\omega e}' \leq B_{\Omega e}(I - T B_{\Omega e}' X_e L_e{}^{-1} B_{\Omega e})^{-1} B_{\Omega e}',$$

from which we conclude that together the last two terms of the expression resulting from (i) are negative semidefinite. Also, since $(X_e L_e{}^{-1})^{-1} - T B_{\omega e} B_{\omega e}' > 0$, then

$$(I - T B_{\omega e}' X_e L_e{}^{-1} B_{\omega e})^{-1} = I + T B_{\omega e}'((X_e L_e{}^{-1})^{-1} - T B_{\omega e} B_{\omega e}')^{-1} B_{\omega e} > 0,$$

and therefore the third to last term is also negative semidefinite. Thus, we satisfy condition (i) of the bounded real lemma by setting

$$\begin{aligned}
0 &= F_e' X_e L_e{}^{-1} + X_e L_e{}^{-1} F_e + T F_e' X_e L_e{}^{-1} F_e + \tfrac{1}{\alpha^2} X_e G_e G_e' X_e L_e{}^{-1} + H_e' H_e \\
&\quad + (I + TF_e)' X_e L_e{}^{-1} B_{\Omega e}(I - T B_{\Omega e}' X_e L_e{}^{-1} B_{\Omega e})^{-1} B_{\Omega e}' X_e L_e{}^{-1}(I + TF_e).
\end{aligned} \tag{A.10}$$

Now consider how this change affects the Riccati equations. Substituting $G_eG_e' + \alpha^2 B_{\Omega e} B_{\Omega e}'$ for G_eG_e' also changes L_e. After some manipulations,

$$0 = F_e'X_eL_e^{-1} + X_eL_e^{-1}F_e + TF_e'X_eL_e^{-1}F_e + \frac{1}{\alpha^2}X_eG_eG_e'X_eL_e^{-1} + H_e'H_e$$

becomes (A.10). Thus, the changes in the theorem derived from (i) can be found by substituting $G_eG_e' + \alpha^2 B_{\Omega e}B_{\Omega e}'$ for G_eG_e'. This is equivalent to substituting $GG' + \alpha^2 B_\Omega B_\Omega{}'$ for GG', as can be seen by expanding

$$G_eG_e' = \begin{pmatrix} GG' & -GG' \\ -GG' & GG' + K^oK^{o\prime} \end{pmatrix}$$

$$G_eG_e' + \alpha^2 B_{\Omega e}B_{\Omega e}' = \begin{pmatrix} GG' + \alpha^2 B_\Omega B_\Omega{}' & -GG' - \alpha^2 B_\Omega B_\Omega{}' \\ -GG' - \alpha^2 B_\Omega B_\Omega{}' & GG' + \alpha^2 B_\Omega B_\Omega{}' + K^oK^{o\prime} \end{pmatrix}.$$

This is equivalent to replacing G everywhere with $(G\ \alpha B_\Omega)$. The changed equations are (2.23), (2.24), and

$$\Lambda := I + T(B_\Omega B_\Omega{}' - \frac{1}{\alpha^2}GG')X,$$

and, in the observer,

$$\begin{aligned} (G\ \alpha B_\Omega)\hat{w}_0 &= \tfrac{1}{\alpha^2}(G\ \alpha B_\Omega)\left(\begin{smallmatrix} G' \\ \alpha B_\Omega{}' \end{smallmatrix}\right)X\Lambda^{-1}(I+TA)\xi \\ &= \tfrac{1}{\alpha^2}GG'X\Lambda^{-1}(I+TA)\xi + B_\Omega B_\Omega{}'X\Lambda^{-1}(I+TA)\xi, \end{aligned}$$

which accounts for the extra term in the observer.

The equivalence, and thus the substitution, was only for condition (i) of the bounded real lemma because its form was found to be equivalent to the standard form with the substitution. We check condition (ii) and detectability next.

Consider condition (ii)

$$\alpha^2 I - TG_e'X_eG_e > 0$$

and our assumption

$$I - TB_{\Omega e}'X_eL_e^{-1}B_{\Omega e} > 0.$$

If $X > 0$, they can be rewritten as

$$\begin{pmatrix} X^{-1} - T\frac{1}{\alpha^2}GG' & T\frac{1}{\alpha^2}GG' \\ T\frac{1}{\alpha^2}GG' & X_1{}^{-1} - T\frac{1}{\alpha^2}(GG' + K^oK^{o\prime}) \end{pmatrix} > 0$$

and

$$\begin{pmatrix} X^{-1} - T\frac{1}{\alpha^2}GG' & T\frac{1}{\alpha^2}GG' \\ T\frac{1}{\alpha^2}GG' & X_1{}^{-1} - T\frac{1}{\alpha^2}(GG' + K^oK^{o\prime}) \end{pmatrix} > \begin{pmatrix} TB_\Omega B_\Omega' & -TB_\Omega B_\Omega' \\ -TB_\Omega B_\Omega' & TB_\Omega B_\Omega' \end{pmatrix}.$$

Since

$$\begin{pmatrix} TB_\Omega B_\Omega' & -TB_\Omega B_\Omega' \\ -TB_\Omega B_\Omega' & TB_\Omega B_\Omega' \end{pmatrix} \geq 0,$$

the latter is more restrictive and will guarantee (ii). But this is equivalent to

$$\begin{pmatrix} X^{-1} - T\frac{1}{\alpha^2}(G\ \alpha B_\Omega)\binom{G'}{\alpha B_\Omega'} & T\frac{1}{\alpha^2}(G\ \alpha B_\Omega)\binom{G'}{\alpha B_\Omega'} \\ T\frac{1}{\alpha^2}(G\ \alpha B_\Omega)\binom{G'}{\alpha B_\Omega'} & X_1{}^{-1} - T\frac{1}{\alpha^2}((G\ \alpha B_\Omega)\binom{G'}{\alpha B_\Omega'} + K^oK^{o\prime}) \end{pmatrix} > 0.$$

This is the same as in the standard problem except that G is replaced, as in condition (i), by $(G\ \alpha B_\Omega)$. But then the conditions guaranteeing this can be found, using (i), by substituting for G in the standard conditions. Thus, we require

$$0 < X^{-1} - T\frac{1}{\alpha^2}GG' - TB_\Omega B_\Omega'$$

(or $0 < I - TB_\Omega{}'XL^{-1}B_\Omega$), and, if $|(L - TB_\Omega B_\Omega{}'X)^{-1}(I+TA) - TK^oC| \neq 0$,

$$\begin{aligned} \alpha^2 Y^{-1} &> (I+TA)'X(L - TB_\Omega B_\Omega{}'X)^{-1}(I+TA) + TH'H \\ &= (I+TA)'XL^{-1}(I+TA) + TH'H \\ &\quad + (I+TA)'XL^{-1}TB_\Omega(I - TB_\Omega{}'XL^{-1}B_\Omega)^{-1}B_\Omega{}'XL^{-1}(I+TA). \end{aligned}$$

Now, check the detectability condition. We require $(\bar{F}_e, \bar{H}_e)$ to be detectable. Suppose $v = \binom{v_1}{v_2} \neq 0$ satisfies

$$\begin{pmatrix} A + BK^c - B_\omega K^c_\omega & BK^c - B_\omega K^c_\omega \\ GK^d + B_\omega K^c_\omega & A + GK^d - K^oC + B_\omega K^c_\omega \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} H & 0 \\ K^c - K^c_\omega & K^c - K^c_\omega \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$$

Then, $Av_1 = \lambda v_1$ and $Hv_1 = 0$. Since (A, H) is detectable, either $v_1 = 0$ or λ is in the stability region. If $v_1 = 0$, then

$$(A + GK^d + B_\omega K^c_\omega - K^oC)v_2 = \lambda v_2$$

and

$$(BK^c - B_\omega K^c_\omega)v_2 = 0,$$

and thus

$$(A + GK^d - K^o C + BK^c)v_2 = \lambda v_2.$$

Thus, λ is in the stability region and $(\bar{F}_e, \bar{H}_e)$ is detectable. □

A.4 Derivations of the Controller Designs for Decentralized Systems

First, the derivation for the H_∞-norm-bounding decentralized controller design of Theorem 2.4.1 is presented.

Theorem For the decentralized system (2.7), (2.8), with (A, H) detectable and with observer-based feedback (2.10), (2.11), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_e z}\|_\infty \leq \alpha$ is

$$K_i^c = -B_i'\mathcal{X}, \quad i \in \{1, 2, ..., p\}, \qquad K^d = \frac{1}{\alpha^2}G'\mathcal{X},$$

where

$$\mathcal{X} := X\Lambda^{-1}(I + TA), \qquad \Lambda := I + T(BB' - \frac{1}{\alpha^2}GG')X, \qquad L := I - T\frac{1}{\alpha^2}GG'X$$

and K_D^o block diagonal, $X \geq 0$, and $W > 0$ satisfy

$$A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(BB' - \frac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H = 0$$

$$\begin{aligned}
&A_f W + WA_f' + TA_f WA_f' + \tfrac{1}{\alpha^2}G_C(I - T\tfrac{1}{\alpha^2}G'XG)^{-1}G_C' + \tfrac{1}{\alpha^2}K_D^o K_D^{o\prime} \\
&\quad + (I + TA_f)W\mathcal{X}_D' B_D\left((I + TB'XL^{-1}B)^{-1} - TB_D'\mathcal{X}_D W\mathcal{X}_D' B_D\right)^{-1} \\
&\qquad \cdot B_D'\mathcal{X}_D W(I + TA_f)' \\
&\quad + TK_D^o C_D(W^{-1} - T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}C_D' K_D^{o\prime} \\
&\quad - K_D^o C_D(W^{-1} - T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}(I + TA_f)' \\
&\quad - (I + TA_f)(W^{-1} - T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}C_D' K_D^{o\prime} = 0
\end{aligned}$$

where

$$A_f(X) := A_e + T\frac{1}{\alpha^2}G_C G'XL^{-1}BB_D'\mathcal{X}_D,$$

such that the eigenvalues of $A_e - K_D^o C_D$ are in the stability region and either

$$\Xi_D(X, W) := \begin{pmatrix} \alpha^2 I - TG'XG - TG_C' W^{-1}G_C & TG_C' W^{-1}K_D^o \\ TK_D^{o\prime}W^{-1}G_C & \alpha^2 I - TK_D^{o\prime}W^{-1}K_D^o \end{pmatrix} > 0$$

or

$$X > 0, \qquad \alpha^2 I - TG'XG > 0, \qquad |I + TA_f - TK_D^o C_D| \neq 0,$$

$$W^{-1} > T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D.$$

Proof We will use the bounded real lemma with the gains (2.25) where $X \geq 0$ satisfies (2.27) and derive the remaining conditions. Let $e_i := \xi_i - x$ and

$$e := \begin{pmatrix} e_1 \\ \vdots \\ e_p \end{pmatrix}.$$

Then the closed-loop system resulting from applying the dynamic feedback (2.10), (2.11), with gains (2.25) to the decentralized system (2.7), (2.8), is

$$\begin{aligned} \rho\begin{pmatrix} x \\ e \end{pmatrix} &= \begin{pmatrix} A - BB'\mathcal{X} & -BB_D'\mathcal{X}_D \\ \frac{1}{\alpha^2}G_C G'\mathcal{X} & A_e - K_D^o C_D \end{pmatrix}\begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} G & 0 \\ -G_C & K_D^o \end{pmatrix}\begin{pmatrix} w_0 \\ w \end{pmatrix} \\ &=: F_e\begin{pmatrix} x \\ e \end{pmatrix} + G_e w_e, \end{aligned}$$

$$z = \begin{pmatrix} H & 0 \\ -B'\mathcal{X} & -B_D'\mathcal{X}_D \end{pmatrix}\begin{pmatrix} x \\ e \end{pmatrix} =: H_e\begin{pmatrix} x \\ e \end{pmatrix}.$$

Applying the bounded real lemma, as before, with $X_e = \left(\begin{smallmatrix} X & 0 \\ 0 & X_1 \end{smallmatrix}\right)$, part (i) has upper-left, upper-right, and lower-left elements zero. The lower-right element of (i) takes the form

$$\begin{aligned} 0 = {} & T(A_e - K_D^o C_D + T\tfrac{1}{\alpha^2}G_C G'XL^{-1}BB_D'\mathcal{X}_D)'X_1 D^{-1} \\ & \cdot (A_e - K_D^o C_D + T\tfrac{1}{\alpha^2}G_C G'XL^{-1}BB_D'\mathcal{X}_D) \\ & + \mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D \\ & + (A_e - K_D^o C_D - T\tfrac{1}{\alpha^2}G_C G'XL^{-1}BB_D'\mathcal{X}_D)'X_1 D^{-1} \\ & + X_1 D^{-1}(A_e - K_D^o C_D + T\tfrac{1}{\alpha^2}G_C G'XL^{-1}BB_D\mathcal{X}_D) \\ & + (\tfrac{1}{\alpha^2}X_1 G_C G'XL^{-1}T\tfrac{1}{\alpha^2}GG_C' + \tfrac{1}{\alpha^2}X_1 G_C G_C' + \tfrac{1}{\alpha^2}X_1 K_D^o K_D^{o\prime})X_1 D^{-1} \end{aligned}$$

where $D := I - T\frac{1}{\alpha^2}K_D^o K_D^{o\prime}X_1 - T\frac{1}{\alpha^2}G_C(I - T\frac{1}{\alpha^2}G'XG)^{-1}G_C'X_1$.

Now consider two cases, $T = 0$ and $T \neq 0$. The case $T = 0$ reduces to

$$\begin{aligned} 0 &= (A_e - K_D^o C_D)'X_1 + X_1(A_e - K_D^o C_D) + X_D' B_D B_D' X_D \\ &\quad + \tfrac{1}{\alpha^2}X_1 G_C G_C' X_1 + \tfrac{1}{\alpha^2}X_1 K_D^o K_D^{o\prime} X_1, \end{aligned}$$

where

$$X_D := \begin{pmatrix} X & 0 & \cdots & 0 \\ 0 & X & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & X \end{pmatrix}.$$

Substituting $K_D^o := \alpha^2 W_D C_D'$ and multiplying on the right and left by $W := X_1{}^{-1}$, we arrive at the design equation from [25]:

$$\begin{aligned} 0 &= WA_e' + A_e W + WX_D' B_D B_D' X_D W - \alpha^2 W C_D' C_D W \\ &\quad + \tfrac{1}{\alpha^2} G_C G_C' + \alpha^2 (W_D - W) C_D' C_D (W_D - W). \end{aligned}$$

This is called a Riccati-like equation in [25] since, if the final term were a constant, it would be a standard Riccati equation. The unified form (2.28) reduces to this form when $T = 0$ and $K_D^o := \alpha^2 W_D C_D'$. The choice made in [25] for W_D was the block diagonal part of W. This was done to "minimize" the nonnegative-definite final term of the Riccati-like algebraic equation.

Now consider the case $T \neq 0$. Multiplying by T and adding and subtracting $X_1 D^{-1}$,

$$\begin{aligned} X_1 &= T\mathcal{X}_D' B_D (I + TB'XL^{-1}B) B_D' \mathcal{X}_D \\ &\quad + [I + T(A_e - K_D^o C_D + T\tfrac{1}{\alpha^2} G_C G' X L^{-1} B B_D' \mathcal{X}_D)]' X_1 D^{-1} \\ &\quad \cdot [I + T(A_e - K_D^o C_D + T\tfrac{1}{\alpha^2} G_C G' X L^{-1} B B_D' \mathcal{X}_D)]. \end{aligned} \tag{A.11}$$

To simplify this expression, define A_f as in (2.29). Then,

$$\begin{aligned} X_1 &= T\mathcal{X}_D' B_D (I + TB'XL^{-1}B) B_D' \mathcal{X}_D \\ &\quad + [(I + TA_f) - TK_D^o C_D]' \\ &\quad \cdot (X_1{}^{-1} - T\tfrac{1}{\alpha^2} G_C (I - T\tfrac{1}{\alpha^2} G' X G)^{-1} G_C' - T\tfrac{1}{\alpha^2} K_D^o K_D^{o\prime})^{-1} \\ &\quad \cdot [(I + TA_f) - TK_D^o C_D]. \end{aligned}$$

Subtracting the constant term and multiplying on either side by $[(I + TA_f) - TK_D^o C_D]^{-1}$, or its transpose, we obtain

$$\begin{aligned} &(X_1{}^{-1} - T\tfrac{1}{\alpha^2} G_C (I - T\tfrac{1}{\alpha^2} G' X G)^{-1} G_C' - T\tfrac{1}{\alpha^2} K_D^o K_D^{o\prime})^{-1} \\ &\quad = [(I + TA_f) - TK_D^o C_D]'^{-1} (X_1 - T\mathcal{X}_D' B_D (I + TB'XL^{-1}B) B_D' \mathcal{X}_D \\ &\quad\quad \cdot [(I + TA_f) - TK_D^o C_D]^{-1}. \end{aligned}$$

Inverting and subtracting the constant term, we find

$$\begin{aligned} X_1{}^{-1} &= [(I + TA_f) - TK_D^o C_D](X_1 - T\mathcal{X}_D' B_D (I + TB'XL^{-1}B) B_D' \mathcal{X}_D)^{-1} \\ &\quad \cdot [(I + TA_f) - TK_D^o C_D]' \\ &\quad + T\tfrac{1}{\alpha^2} G_C (I - T\tfrac{1}{\alpha^2} G' X G)^{-1} G_C' + T\tfrac{1}{\alpha^2} K_D^o K_D^{o\prime}. \end{aligned}$$

Substituting $W := X_1{}^{-1}$, we arrive at

$$\begin{aligned} W = & [(I + TA_f) - TK_D^{\circ}C_D](W^{-1} - T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1} \\ & \cdot [(I + TA_f) - TK_D^{\circ}C_D]' \\ & + T\tfrac{1}{\alpha^2}G_C(I - T\tfrac{1}{\alpha^2}G'XG)^{-1}G_C' + T\tfrac{1}{\alpha^2}K_D^{\circ}K_D^{\circ\prime}. \end{aligned} \tag{A.12}$$

Rearranging the terms reduces the above equation to (2.28). We arrange it this way to parallel the associated difference equation that converges numerically in our examples to the desired solution, as discussed in Section 2.6.

The substitution for K_D° corresponding to the substitution in the continuous-time case would be $K_D^{\circ} := \alpha^2 W_D C_D'$, where W_D is the block diagonal part of W. From working examples, we find that this choice results in minimal H_∞ norms close to those obtained for the continuous-time decentralized system.

Now, part (ii) of the bounded real lemma can be expanded directly as (2.30). If, further, we assume that $X > 0$, then $\alpha^2 I - TG_e'X_eG_e > 0$ is equivalent to $X_e{}^{-1} - T\frac{1}{\alpha^2}G_eG_e' > 0$, which can be expanded as

$$\begin{pmatrix} I & 0 \\ T\frac{1}{\alpha^2}G_CG'XL^{-1} & I \end{pmatrix} \begin{pmatrix} LX^{-1} & 0 \\ 0 & DX_1{}^{-1} \end{pmatrix} \begin{pmatrix} I & T\frac{1}{\alpha^2}XL^{-1}GG_C' \\ 0 & I \end{pmatrix} > 0$$

or to

$$\alpha^2 I - TG'XG > 0$$

and, by (A.11), as long as $|I + TA_f - TK_D^{\circ}C_D| \neq 0$, to

$$X_1 - T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D > 0,$$

or to

$$W^{-1} - T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D > 0.$$

Now, we check the detectability condition. We assume that (A, H) is detectable and that the eigenvalues of $A_e - K_D^{\circ}C_D$ are in the stability region D_T and check the detectability of

$$(F_e, H_e) = \left(\begin{pmatrix} A - BB'\mathcal{X} & -BB_D'\mathcal{X}_D \\ \frac{1}{\alpha^2}G_CG'\mathcal{X} & A_e - K_D^{\circ}C_D \end{pmatrix}, \begin{pmatrix} H & 0 \\ -B'\mathcal{X} & -B_D'\mathcal{X}_D \end{pmatrix} \right).$$

Let $v = \binom{v_1}{v_2} \neq 0$ satisfy $F_ev = \lambda v$ and $H_ev = 0$. Then

$$Av_1 - B(B'\mathcal{X}v_1 + B_D'\mathcal{X}_Dv_2) = \lambda v_1,$$

$$-(B'\mathcal{X}v_1 + B_D'\mathcal{X}_D v_2) = 0, \quad \text{and} \quad Hv_1 = 0,$$

and thus $Av_1 = \lambda v_1$ and $Hv_1 = 0$. Since (A, H) is detectable, either $\lambda \in D_T$, or $v_1 = 0$. If $v_1 = 0$, then $(A_e - K_D^o C_D)v_2 = \lambda v_2$. But then $\lambda \in D_T$. □

Next, the derivation of the sensor-outage reliable controller design of Theorem 2.4.2 is presented.

Theorem For the decentralized system (2.7), (2.8), with (A, H) detectable and with observer-based feedback (2.10), (2.11), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_e z}\|_\infty \leq \alpha$ for all subsystem sensor failures $\omega \subseteq \Omega$ is

$$K_i^c = -B_i'\mathcal{X}, \quad i \in \{1, 2, ..., p\}, \qquad K^d = \frac{1}{\alpha^2} G'\mathcal{X},$$

where

$$\Lambda := I + T(BB' - \frac{1}{\alpha^2}GG')X, \qquad L := I - T\frac{1}{\alpha^2}GG'X$$

and K_D^o block diagonal, $X \geq 0$, and $W > 0$ satisfy

$$\begin{aligned}
&A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(BB' - \tfrac{1}{\alpha^2}GG')X\Lambda^{-1} \\
&\quad + H'H + \alpha^2 C_\Omega' C_\Omega = 0
\end{aligned}$$

$$\begin{aligned}
&A_f W + WA_f' + TA_f W A_f' + \tfrac{1}{\alpha^2}G_C(I - T\tfrac{1}{\alpha^2}G'XG)^{-1}G_C' + \tfrac{1}{\alpha^2}K_D^o K_D^{o\prime} \\
&\quad + (I + TA_f)W\mathcal{X}_D' B_D\left((I + TB'XL^{-1}B)^{-1} - TB_D'\mathcal{X}_D W \mathcal{X}_D' B_D\right)^{-1} \\
&\qquad \cdot B_D'\mathcal{X}_D W(I + TA_f)' \\
&\quad + TK_D^o C_D(W^{-1} - T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}C_D' K_D^{o\prime} \\
&\quad - K_D^o C_D(W^{-1} - T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}(I + TA_f)' \\
&\quad - (I + TA_f)(W^{-1} - T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}C_D' K_D^{o\prime} = 0
\end{aligned}$$

where

$$A_f(X) := A_e + T\frac{1}{\alpha^2}G_C G'XL^{-1}BB_D'\mathcal{X}_D,$$

such that the eigenvalues of $A_e - K_D^o C_D$ are in the stability region and either both $\Xi_D(X, W) > 0$ and, $\forall \omega \subseteq \Omega$,

$$\bar{\Xi}_D(X, W) := \begin{pmatrix} \alpha^2 I - TG'XG - TG_C' W^{-1} G_C & TG_C' W^{-1} K_{D,\bar{\omega}}^o \\ TK_{D,\bar{\omega}}^{o\prime} W^{-1} G_C & \alpha^2 I - TK_{D,\bar{\omega}}^{o\prime} W^{-1} K_{D,\bar{\omega}}^o \end{pmatrix} > 0$$

or

$$X > 0, \qquad \alpha^2 I - TG'XG > 0, \qquad |I + TA_f - TK_D^o C_D| \neq 0,$$

$$W^{-1} > T\mathcal{X}_D' B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D.$$

Proof Since $y_i = 0 \ \forall i \in \omega$, the closed-loop system with outages has changes only in the $K^o{}_i y_i$ terms in the observers. The resulting closed-loop system matrices are

$$\bar{F}_e = \begin{pmatrix} A - BB'\mathcal{X} & -BB'_D\mathcal{X}_D \\ \frac{1}{\alpha^2}G_C G'\mathcal{X} - K^o_{D,\omega}C_\omega & A_e - K^o_D C_D \end{pmatrix}, \qquad \bar{G}_e = \begin{pmatrix} G & 0 \\ -G_C & K^o_{D,\bar{\omega}} \end{pmatrix},$$

$$\bar{H}_e = \begin{pmatrix} H & 0 \\ -B'\mathcal{X} & -B'_D\mathcal{X}_D \end{pmatrix}.$$

We want to find conditions to guarantee that, for all $\omega \subseteq \Omega$, the closed-loop system will be stable and $\|T_{w_e z}\| \leq \alpha$.

Note that the form of the changes to the closed-loop system matrices is the same as in the sensor-outage centralized case, i.e.,

$$\bar{F}_e = F_e - K^o_{\omega e}C_{\omega e}, \qquad \bar{G}_e\bar{G}'_e = G_e G'_e - K^o_{\omega e}K^{o\prime}_{\omega e}, \qquad \bar{H}_e = H_e,$$

except, in this case,

$$K^o_{\omega e} := \begin{pmatrix} 0 \\ K^o_{D,\omega} \end{pmatrix}, \qquad C_{\omega e} := (C_\omega \ 0).$$

Thus, following the development of the sensor-outage centralized case, if $\alpha^2 I - TG'_e X_e G_e > 0$, the conditions that guarantee condition (i) of the bounded real lemma can be found by replacing H with $\binom{H}{\alpha C_\Omega}$ in the conditions derived from (i). Thus, the change is only in (2.33), since H appears only in that condition.

Next, we check when condition (ii) of the bounded real lemma,

$$\alpha^2 I - T\bar{G}'_e X_e \bar{G}_e > 0,$$

and

$$\alpha^2 I - TG'_e X_e G_e > 0,$$

which is required from condition (i), hold for all $\omega \subseteq \Omega$. This requires that $\Xi_D(X, W) > 0$ and $\bar{\Xi}_D(X, W) > 0 \ \forall \omega \subseteq \Omega$.

If we assume further that $X > 0$, these two conditions reduce to $LX^{-1} > 0$, $DX_1{}^{-1} > 0$, and $DX_1{}^{-1} + T\frac{1}{\alpha^2}K^o_{D,\omega}K^{o\prime}_{D,\omega} > 0$. The third of these inequalities holds whenever the second inequality holds, but the first two inequalities are the same as in the basic decentralized case. Since H does not appear in (A.11), the resulting conditions are the same as in the basic decentralized theorem.

Finally, there are no changes in the conditions resulting from the detectability condition since the change in $\bar{F}_e$ appears only in the lower-left element. □

Next, the derivation of the actuator-outage reliable controller design of Theorem 2.4.3 is given.

Theorem For the decentralized system (2.7), (2.8), with (A, H) detectable and with observer-based feedback

$$\rho\xi_i = A\xi_i + \sum_{j\in\bar{\Omega}} B_j\hat{u}_j^i + G\hat{w}_0^i + K^o{}_i(y_i - C_i\xi_i)$$

$$u_i = K_i^c\xi_i, \qquad \hat{u}_j^i = K_j^c\xi_i, \qquad \hat{w}_0^i = K^d\xi_i,$$

for $i \in \{1, 2, ..., p\}$, a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{w_e z}\|_\infty \leq \alpha$ for all $\omega \subseteq \Omega$ is

$$K_i^c = -B_i'\mathcal{X}, \quad i \in \{1, 2, ..., p\}, \qquad K^d = \frac{1}{\alpha^2}G'\mathcal{X},$$

where

$$\Lambda := I + T(B_{\bar{\Omega}}B_{\bar{\Omega}}' - \frac{1}{\alpha^2}GG')X, \qquad L := I - T\frac{1}{\alpha^2}GG'X$$

and K_D^o block diagonal, $X > 0$, and $W > 0$ satisfy

$$A'X\Lambda^{-1} + X\Lambda^{-1}A + TA'X\Lambda^{-1}A - X(B_{\bar{\Omega}}B_{\bar{\Omega}}' - \frac{1}{\alpha^2}GG')X\Lambda^{-1} + H'H = 0$$

$$\begin{aligned}
&A_fW + WA_f' + TA_fWA_f' + \tfrac{1}{\alpha^2}K_D^oK_D^{o\prime} \\
&\quad + \tfrac{1}{\alpha^2}(G_C\ \alpha B_{C,\Omega})(I - T\tfrac{1}{\alpha^2}\left(\begin{smallmatrix} G' \\ \alpha B_\Omega' \end{smallmatrix}\right)X(G\ \alpha B_\Omega))^{-1}\left(\begin{smallmatrix} G_C' \\ \alpha B_{C,\Omega}' \end{smallmatrix}\right) \\
&\quad + (I + TA_f)W\mathcal{X}_D'B_D\left((I + TB'XL^{-1}B)^{-1} - TB_D'\mathcal{X}_DW\mathcal{X}_D'B_D\right)^{-1} \\
&\quad\quad \cdot B_D'\mathcal{X}_DW(I + TA_f)' \\
&\quad + TK_D^oC_D(W^{-1} - T\mathcal{X}_D'B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}C_D'K_D^{o\prime} \\
&\quad - K_D^oC_D(W^{-1} - T\mathcal{X}_D'B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}(I + TA_f)' \\
&\quad - (I + TA_f)(W^{-1} - T\mathcal{X}_D'B_D(I + TB'XL^{-1}B)B_D'\mathcal{X}_D)^{-1}C_D'K_D^{o\prime} = 0
\end{aligned}$$

where

$$\begin{aligned}
A_f(X) \quad := \quad & A_e + T\tfrac{1}{\alpha^2}(G_C\ \alpha B_{C,\Omega})\left(\begin{smallmatrix} G' \\ \alpha B_\Omega' \end{smallmatrix}\right)X(L - TB_\Omega B_\Omega'X)^{-1}BB_D'\mathcal{X}_D \\
& + B_{D,\Omega}B_{D,\Omega}'\mathcal{X}_D + TB_{C,\Omega}B_\Omega'X(L - TB_\Omega B_\Omega'X)^{-1}BB_D'\mathcal{X}_D
\end{aligned}$$

such that the eigenvalues of $A_e - B_CB_D'\mathcal{X}_D - K_D^oC_D$ are in D_T and

$$I - T(B_\Omega' \ \ - B_{C,\Omega}')X_eL_e{}^{-1}\begin{pmatrix} B_\Omega \\ -B_{C,\Omega} \end{pmatrix} > 0$$

or

$$X^{-1} - T\frac{1}{\alpha^2}GG' - TB_\Omega B_\Omega' > 0, \qquad |I + TA_f - TK_D^oC_D| \neq 0,$$

$$W^{-1} > T\mathcal{X}_D'B_D[I + TB'X(L - TB_\Omega B_\Omega'X)^{-1}B]B_D'\mathcal{X}_D.$$

Proof Suppose we start with the standard observer form (2.10). If the failures occur as $u_i = 0\ \forall i \in \omega$, the changes occur in the state equation and the observer row of the output matrix. The resulting closed-loop system matrices are

$$\bar{F}_e = \begin{pmatrix} A - BB'\mathcal{X} + B_\omega B'_\omega \mathcal{X} & -BB'_D\mathcal{X}_D + B_\omega B'_{D,\omega}\mathcal{X}_D \\ \frac{1}{\alpha^2}G_C G'\mathcal{X} - B_{C,\omega}B'_\omega\mathcal{X} & A_e - K^o_D C_D - B_{C,\omega}B'_{D,\omega}\mathcal{X}_D \end{pmatrix},$$

$$\bar{G}_e = \begin{pmatrix} G & 0 \\ -G_C & K^o_D \end{pmatrix}, \qquad \bar{H}_e = \begin{pmatrix} H & 0 \\ -B'_{\bar{\omega}}\mathcal{X} & -B'_{D,\bar{\omega}}\mathcal{X}_D \end{pmatrix}.$$

The relationship between the system with failures and the system without failures can thus be expressed as

$$\bar{F}_e = F_e - B_{\omega e}K^c_{\omega e}, \qquad \bar{G}_e = G_e,$$

$$\bar{H}'_e\bar{H}_e = H'_eH_e - K^{c\prime}_{\omega e}K^c_{\omega e},$$

where, in this case,

$$B_{\omega e} := \begin{pmatrix} B_\omega \\ -B_{C,\omega} \end{pmatrix}, \qquad K^c_{\omega e} := \begin{pmatrix} -B'_\omega\mathcal{X} & -B'_{D,\omega}\mathcal{X}_D \end{pmatrix}.$$

The form of the changes is the same as that in the centralized actuator-outage case. Therefore, if $X > 0$ and $I - TB'_{\Omega e}X_eL_e{}^{-1}B_{\Omega e} > 0$, the changes in the conditions that guarantee (i) in the bounded real lemma can be found by replacing $G_eG'_e$ by $G_eG'_e + \alpha^2 B_{\Omega e}B'_{\Omega e}$. But

$$G_eG'_e = \begin{pmatrix} GG' & -GG'_C \\ -G_CG' & G_CG'_C + K^o_DK^{o\prime}_D \end{pmatrix}$$

and

$$G_eG'_e + \alpha^2 B_{\Omega e}B'_{\Omega e} = \begin{pmatrix} GG' + \alpha^2 B_\Omega B'_\Omega & -(GG'_C + \alpha^2 B_\Omega B'_{C,\Omega}) \\ -(G_CG' + \alpha^2 B_{C,\Omega}B'_\Omega) & G_CG'_C + \alpha^2 B_{C,\Omega}B'_{C,\Omega} + K^o_DK^{o\prime}_D \end{pmatrix}.$$

Note that, if we change from the basic observer to one with an extra term corresponding to αB_Ω, this is equivalent to substituting $(G\ \alpha B_\Omega)$ for G and $(G_C\ \alpha B_{C,\Omega})$ for G_C in the conditions guaranteeing part (i) of the bounded real lemma in the basic decentralized case.

The modified conditions are thus (2.39), (2.40),

$$\Lambda := I + T(B_{\bar{\Omega}}B'_{\bar{\Omega}} - \frac{1}{\alpha^2}GG')X,$$

and (2.41) where the $B_{D,\Omega}B'_{D,\Omega}\mathcal{X}_D$ term is a result of the presence of G in A_e.

The change in the observer is

$$\begin{aligned}(G\ \alpha B_\Omega)\hat{w}_0^i &= \tfrac{1}{\alpha^2}(G\ \alpha B_\Omega)\binom{G'}{\alpha B'_\Omega} X\Lambda^{-1}(I+TA)\xi_i \\ &= \tfrac{1}{\alpha^2}GG'\mathcal{X}\xi_i + B_\Omega B'_\Omega \mathcal{X}\xi_i.\end{aligned}$$

Thus, the new observer is (2.37).

Now consider condition (ii), $\alpha^2 I - T\bar{G}'_e X_e \bar{G}_e > 0$, and $I - TB'_{\Omega e}X_e L_e{}^{-1}B_{\Omega e} > 0$. If $X > 0$, they can be rewritten as

$$\begin{pmatrix} X^{-1} - T\frac{1}{\alpha^2}GG' & T\frac{1}{\alpha^2}GG'_C \\ T\frac{1}{\alpha^2}G_CG' & X_1{}^{-1} - T\frac{1}{\alpha^2}(G_CG'_C + K_D^oK_D^{o\prime}) \end{pmatrix} > 0$$

and

$$\begin{pmatrix} X^{-1} - T\frac{1}{\alpha^2}GG' & T\frac{1}{\alpha^2}GG'_C \\ T\frac{1}{\alpha^2}G_CG' & X_1{}^{-1} - T\frac{1}{\alpha^2}(G_CG'_C + K_D^oK_D^{o\prime}) \end{pmatrix} > \begin{pmatrix} TB_\Omega B'_\Omega & -TB_\Omega B'_{C,\Omega} \\ -TB_{C,\Omega}B'_\Omega & TB_{C,\Omega}B'_{C,\Omega} \end{pmatrix}.$$

Since

$$\begin{pmatrix} TB_\Omega B'_\Omega & -TB_\Omega B'_{C,\Omega} \\ -TB_{C,\Omega}B'_\Omega & TB_{C,\Omega}B'_{C,\Omega} \end{pmatrix} \geq 0,$$

the latter is more restrictive and will guarantee (ii). But this is equivalent to

$$\begin{pmatrix} X^{-1} - T\frac{1}{\alpha^2}(G\ \alpha B_\Omega)\binom{G'}{\alpha B'_\Omega} & T\frac{1}{\alpha^2}(G\ \alpha B_\Omega)\binom{G'_C}{\alpha B'_{C,\Omega}} \\ T\frac{1}{\alpha^2}(G_C\ \alpha B_{C,\Omega})\binom{G'}{\alpha B'_\Omega} & X_1{}^{-1} - T\frac{1}{\alpha^2}\left((G_C\ \alpha B_{C,\Omega})\binom{G'_C}{\alpha B'_{C,\Omega}} + K_D^oK_D^{o\prime}\right) \end{pmatrix} > 0.$$

This is the same form as in the standard decentralized problem except that G is replaced by $(G\ \alpha B_\Omega)$ and G_C is replaced by $(G_C\ \alpha B_{C,\Omega})$, as in (i).

Thus, we obtain the conditions at the end of the theorem.

Now check the detectability condition in the bounded real lemma—that $(\bar{F}_e, \bar{H}_e)$ is detectable. Let $\binom{v_1}{v_2}$ be an eigenvector of $\bar{F}_e$ in the null space of $\bar{H}_e$. Then,

$$\begin{pmatrix} A - BB'\mathcal{X} + B_\omega B'_\omega \mathcal{X} & -BB'_D\mathcal{X}_D + B_\omega B'_{D,\omega}\mathcal{X}_D \\ \frac{1}{\alpha^2}G_CG'\mathcal{X} - B_{C,\omega}B'_\omega\mathcal{X} & A_e - K_D^oC_D - B_{C,\omega}B'_{D,\omega}\mathcal{X}_D \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} H & 0 \\ -B'\mathcal{X} + B'_\omega\mathcal{X} & -B'_D\mathcal{X}_D + B'_{D,\omega}\mathcal{X}_D \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$$

Then $Av_1 = \lambda v_1$ and $Hv_1 = 0$. Since (A, H) is detectable, either $v_1 = 0$ or $\lambda \in D_T$. If $v_1 = 0$, then

$$(A_e - K_D^oC_D - B_{C,\omega}B'_{D,\omega}\mathcal{X}_D)v_2 = \lambda v_2$$

and

$$(B_D'\mathcal{X}_D - B_{D,\omega}'\mathcal{X}_D)v_2 = 0,$$

and thus

$$(A_e - K_D^o C_D - B_C B_D' \mathcal{X}_D)v_2 = \lambda v_2.$$

Thus, $\lambda \in D_T$, so $(\bar{F}_e, \bar{H}_e)$ is detectable. □

APPENDIX B

PROOFS OF THEOREMS AND LEMMAS IN CHAPTER 3

The proofs of Lemmas 3.3.1 and 3.3.2 and Theorems 3.3.1 and 3.3.3 from Chapter 3 are provided here.

B.1 Proofs of Generalized Bounded Real Lemmas

First, the proof is presented for Lemma 3.3.1.

Lemma Consider a linear system T_{wz} with a detectable realization

$$\rho x = Fx + Gw, \qquad x(0) = 0,$$

$$z = Hx + Ew.$$

If there exist a real, symmetric matrix $X \geq 0$ and a real $\alpha > 0$ such that

(i) $F'X + XF + TF'XF + H'H$
$+ (H'E + (I+TF)'XG)(\alpha^2 I - TG'XG - E'E)^{-1}(E'H + G'X(I+TF)) \leq 0$

(ii) $\alpha^2 I - TG'XG - E'E > 0$

then

(a) the eigenvalues of F lie in D_T, the stability region for sampling interval T

(b) $\|T_{wz}\|_\infty \leq \alpha$.

Proof

(a): Let $v \neq 0$ be an eigenvector of F satisfying $Fv = \lambda v$. From (i),

$$\begin{aligned} 0 \;\geq\; & v^*(F'X + XF + TF'XF + H'H \\ & + (H'E + (I+TF)'XG)(\alpha^2 I - TG'XG - E'E)^{-1}(E'H + G'X(I+TF)))v \\ \\ = \;& (2\Re(\lambda) + T|\lambda|^2)\, v^*Xv + v^*H'Hv \\ & + v^*(H'E + (I+TF)'XG)(\alpha^2 I - TG'XG - E'E)^{-1}(E'H + G'X(I+TF))v. \end{aligned}$$

From (ii), each term is positive semidefinite, with the possible exception of the first. Thus, the first term is negative semidefinite:

$$\left(2\Re(\lambda) + T|\lambda|^2\right) v^*Xv \leq 0.$$

If $(2\Re(\lambda)+T|\lambda|^2)\, v^*Xv < 0$, then $v^*Xv > 0$ and $2\Re(\lambda)+T|\lambda|^2 < 0$. In that case, λ is in the stability region. If $(2\Re(\lambda)+T|\lambda|^2)\, v^*Xv = 0$, then all of the terms must be zero, and therefore $Hv = 0$. Since (F,H) is detectable, $\lambda \in D_T$, which proves (a).

(b): Showing that

$$\|T_{wz}\|_\infty \le \alpha$$

is equivalent to showing that

$$V := \|z\|_2{}^2 - \alpha^2\|w\|_2{}^2 \le 0, \quad \forall w \in \mathcal{L}_2.$$

First note that

$$\mathcal{S}_{t=0}^{\infty} \frac{x(t+T)'Xx(t+T) - x(t)'Xx(t)}{T}\, dt = -x(0)'Xx(0) = 0.$$

(In the continuous-time case, the integral is equal to

$$\lim_{t\to\infty} x(t)'Xx(t) - x(0)'Xx(0),$$

the first term of which is 0 since, by (a), the system is stable. In the discrete-time case, the partial sums are

$$x(kT)'Xx(kT) - x(0)'Xx(0).$$

These converge to $-x(0)'Xx(0)$ since the system is stable.)

Now, we show that $V \le 0$.

$$\begin{aligned}
V &= \|z\|_2{}^2 - \alpha^2\|w\|_2{}^2 - x(0)'Xx(0) \\
&= \mathcal{S}_{t=0}^{\infty}[z(t)'z(t) - \alpha^2 w(t)'w(t) \\
&\qquad + \tfrac{1}{T}(x(t+T)'Xx(t+T) - x(t)'Xx(t))]\, dt \\
&= \mathcal{S}_{t=0}^{\infty}[x(t)'\left(H'H + \tfrac{1}{T}(I+TF)'X(I+TF) - \tfrac{1}{T}X\right)x(t) \\
&\qquad + 2x(t)'(H'E + (I+TF)'XG)\,w(t) + w(t)'(E'E + TG'XG - \alpha^2 I)\,w(t)]\, dt \\
&= \mathcal{S}_{t=0}^{\infty}[x(t)'(H'H + F'X + XF + TF'XF)\,x(t) \\
&\qquad + 2x(t)'(H'E + (I+TF)'XG)\,w(t) + w(t)'(E'E + TG'XG - \alpha^2 I)\,w(t)]\, dt.
\end{aligned}$$

But, by (i),

$$F'X+XF+TF'XF+H'H+(H'E+(I+TF)'XG)(\alpha^2I-TG'XG-E'E)^{-1}(E'H+G'X(I+TF)) \le 0.$$

Therefore,

$$V \le \mathcal{S}_{t=0}^{\infty}[x(t)' (-(H'E+(I+TF)'XG)(\alpha^2I-TG'XG-E'E)^{-1}(E'H+G'X(I+TF)))\, x(t)$$
$$+ 2x(t)' (H'E+(I+TF)'XG)\, w(t) + w(t)' (-(\alpha^2I-TG'XG-E'E))\, w(t)]\, dt$$

$$= -\mathcal{S}_{t=0}^{\infty} [w(t) - (\alpha^2I-TG'XG-E'E)^{-1}(G'X(I+TF)+E'H)x(t)]'$$
$$\cdot (\alpha^2I-TG'XG-E'E)\, [w(t) - (\alpha^2I-TG'XG-E'E)^{-1}(G'X(I+TF)+E'H)x(t)]\ dt$$

$$\le 0, \qquad \text{by (ii)}.$$

□

Next, the proof is provided for Lemma 3.3.2.

Lemma Consider a linear system T_{wz} with a realization

$$\rho x = Fx + Gw, \qquad x(0) = 0,$$

$$z = Hx + Ew$$

with all unobservable modes of (F, H) in D_T^η. If there exist a real, symmetric matrix $X \ge 0$ and a real $\alpha > 0$ such that

(i) $F'X + XF + TF'XF + H'H$
$+ (H'E + (I + TF)'XG)(\alpha^2I - TG'XG - E'E)^{-1}(E'H + G'X(I + TF)) \le -(2 - T\eta)\eta X$

(ii) $\alpha^2I - TG'XG - E'E > 0$

then

(a) the eigenvalues of F lie in D_T^η

(b) $\|T_{wz}\|_\infty \le \alpha$.

Proof

(a): Let $v \ne 0$ be an eigenvector of F satisfying $Fv = \lambda v$. From (i),

$$0 \ge v^*(F'X + XF + TF'XF + H'H + (2 - T\eta)\eta X$$
$$+ (H'E + (I + TF)'XG)(\alpha^2I - TG'XG - E'E)^{-1}(E'H + G'X(I + TF)))v$$

$$= (2\Re(\lambda) + T|\lambda|^2 + (2 - T\eta)\eta)\, v^*Xv + v^*H'Hv$$
$$+ v^*(H'E + (I + TF)'XG)(\alpha^2I - TG'XG - E'E)^{-1}(E'H + G'X(I + TF))v.$$

From (ii), each term is positive semidefinite, with the possible exception of the first. Thus, the first term is negative semidefinite:

$$\left(2\Re(\lambda) + T|\lambda|^2 + (2 - T\eta)\eta\right) v^* X v \leq 0.$$

If $(2\Re(\lambda) + T|\lambda|^2 + (2 - T\eta)\eta)\, v^* X v < 0$, then $v^* X v > 0$ and $2\Re(\lambda) + T|\lambda|^2 + (2 - T\eta)\eta < 0$. Then, $\frac{T}{2}|\gamma|^2 + \Re\gamma < -\eta + \frac{T}{2}\eta^2$. In that case, λ is in D_T^η. If $(2\Re(\lambda) + T|\lambda|^2 + (2 - T\eta)\eta)\, v^* X v = 0$, then all of the terms must be zero, and therefore $Hv = 0$. Thus, v is an unobservable mode of (F, H), and hence $\lambda \in D_T^\eta$, which proves (a).

b: Part (b) is identical to the proof of part (b) of Lemma 3.3.1 since

$$\begin{aligned} &F'X + XF + TF'XF + H'H + (H'E + (I+TF)'XG)(\alpha^2 I - TG'XG - E'E)^{-1}(E'H + G'X(I+TF)) \\ &\quad \leq -(2 - T\eta)\eta X \leq 0. \end{aligned}$$

□

B.2 Derivation of State-feedback Controller for Sampled-data Two-rate Problem

Next, the derivation of the controller in Theorem 3.3.1 is presented.

Theorem For the discrete-time plant

$$\delta x = A_\delta x + B_\delta u + (\sqrt{N} G_\delta)\tilde{w}, \quad x(0) = 0,$$

with the performance-output variable

$$\tilde{z} = \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta x + D_{BH} u) + D_{GH}\tilde{w} \\ u \end{pmatrix},$$

with (A_δ, H_δ) detectable and with state feedback $u = K^c x$, a sufficient condition to guarantee that the closed-loop discrete-time system is stable and has H_∞ norm from $\tilde{w}$ to $\tilde{z}$ less than a given value α is

$$\begin{aligned} K^c = -[&I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ &+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\ \cdot [&B_\delta' X(I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\ &+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X(I + TA_\delta)\right)], \end{aligned}$$

where

$$\Upsilon := \alpha^2 I - TNG_\delta' X G_\delta - D_{GH}' D_{GH},$$

and $X > 0$ satisfies

$$\begin{aligned}
0 = {} & A_\delta' X + X A_\delta + T A_\delta' X A_\delta + \tfrac{1}{N} H_\delta' H_\delta \\
& + \left(\tfrac{1}{\sqrt{N}} H_\delta' D_{GH} + \sqrt{N}(I + TA_\delta)' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right) \\
& - [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\
& \quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]' \\
& \cdot [I + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\
& \quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
& \cdot [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\
& \quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]
\end{aligned}$$

and

$$X^{-1} - TNG_\delta(\alpha^2 I - D_{GH}' D_{GH})^{-1} G_\delta' > 0.$$

Proof For Problem 3.2.2, we choose $u = K^c x$ and solve for the closed-loop system matrices F_e, G_e, H_e, and E_e. The closed-loop system is of the form

$$\delta x = (A_\delta + B_\delta K^c) x + (\sqrt{N} G_\delta)\tilde{w} =: F_e x + G_e \tilde{w},$$

$$\tilde{z} = \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta + D_{BH} K^c) \\ K^c \end{pmatrix} x + \begin{pmatrix} D_{GH} \\ 0 \end{pmatrix} \tilde{w} =: H_e x + E_e \tilde{w}.$$

Requiring condition (i) of the bounded real lemma to hold with equality, we obtain

$$\begin{aligned}
0 \;=\; & (A_\delta + B_\delta K^c)' X + X (A_\delta + B_\delta K^c) + T (A_\delta + B_\delta K^c)' X (A_\delta + B_\delta K^c) + K^{c\prime} K^c \\
& + \tfrac{1}{N}(H_\delta + D_{BH} K^c)'(H_\delta + D_{BH} K^c) \\
& + \left(\tfrac{1}{\sqrt{N}}(H_\delta + D_{BH} K^c)' D_{GH} + \sqrt{N}(I + T(A_\delta + B_\delta K^c))' X G_\delta\right) \\
& \quad \cdot (\alpha^2 I - NTG_\delta' X G_\delta - D_{GH}' D_{GH})^{-1} \\
& \quad \cdot \left(\tfrac{1}{\sqrt{N}} D_{GH}'(H_\delta + D_{BH} K^c) + \sqrt{N} G_\delta' X (I + T(A_\delta + B_\delta K^c))\right) \\[2ex]
=\; & A_\delta' X + X A_\delta + T A_\delta' X A_\delta + \tfrac{1}{N} H_\delta' H_\delta + \left(\tfrac{1}{\sqrt{N}} H_\delta' D_{GH} + \sqrt{N}(I + TA_\delta)' X G_\delta\right) \\
& \quad \cdot (\alpha^2 I - NTG_\delta' X G_\delta - D_{GH}' D_{GH})^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right) \\
& + K^{c\prime}[B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
& \quad \cdot (\alpha^2 I - NTG_\delta' X G_\delta - D_{GH}' D_{GH})^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)] \\
& + [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
& \quad \cdot (\alpha^2 I - NTG_\delta' X G_\delta - D_{GH}' D_{GH})^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]' K^c \\
& + K^{c\prime}[I + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
& \quad \cdot (\alpha^2 I - NTG_\delta' X G_\delta - D_{GH}' D_{GH})^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)] K^c,
\end{aligned}$$

or

$$
\begin{aligned}
0 \;=\;& A_\delta' X + X A_\delta + T A_\delta' X A_\delta + \tfrac{1}{N} H_\delta' H_\delta + \left(\tfrac{1}{\sqrt{N}} H_\delta' D_{GH} + \sqrt{N}(I + T A_\delta)' X G_\delta\right) \\
&\quad \cdot \left(\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH}\right)^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right) \\
&- [B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
&\quad \cdot \left(\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH}\right)^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right)]' \\
&\cdot (I + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
&\quad \cdot \left(\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH}\right)^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right))^{-1} \\
&\cdot [B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
&\quad \cdot \left(\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH}\right)^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right)] \\
&+ \{K^c + [I + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
&\quad \cdot \left(\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH}\right)^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
&\cdot [B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
&\quad \cdot \left(\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH}\right)^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right)]\}' \\
&\cdot \{I + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
&\quad \left(\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH}\right)^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)\} \\
&\cdot \{K^c + [I + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
&\quad \cdot \left(\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH}\right)^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
&\cdot [B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
&\quad \cdot \left(\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH}\right)^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right)]\}.
\end{aligned}
$$

The solution to this can be found by setting

$$
\begin{aligned}
K^c = & -[I + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
&\quad \cdot \left(\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH}\right)^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
&\cdot [B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
&\quad \cdot \left(\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH}\right)^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right)],
\end{aligned}
$$

where $X \geq 0$ satisfies

$$\begin{aligned}
0 = {} & A_\delta' X + X A_\delta + T A_\delta' X A_\delta + \tfrac{1}{N} H_\delta' H_\delta + \left(\tfrac{1}{\sqrt{N}} H_\delta' D_{GH} + \sqrt{N}(I + T A_\delta)' X G_\delta\right) \\
& \cdot (\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH})^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right) \\
& - [B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
& \cdot (\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH})^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right)]' \\
& \cdot (I + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
& \cdot (\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH})^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right))^{-1} \\
& \cdot [B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \\
& \cdot (\alpha^2 I - N T G_\delta' X G_\delta - D_{GH}' D_{GH})^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right)].
\end{aligned}$$

Condition (ii) of the bounded real lemma reduces to $\alpha^2 I - T N G_\delta' X G_\delta - D_{GH}' D_{GH} > 0$. As N increases, so does the dimension of this matrix. However, if $X > 0$, then this condition is equivalent to

$$X^{-1} - T N G_\delta (\alpha^2 I - D_{GH}' D_{GH})^{-1} G_\delta' > 0.$$

To find conditions that guarantee detectability of the closed-loop system, let λ, v, be an eigenvalue-eigenvector pair for F_e, H_e, such that

$$(A_\delta + B_\delta K^c) v = \lambda v,$$

$$\begin{pmatrix} \frac{1}{\sqrt{N}} (H_\delta + D_{BH} K^c) \\ K^c \end{pmatrix} v = 0.$$

Note that $A_\delta v = 0$ and $\frac{1}{\sqrt{N}} H_\delta v = 0$. For all integers $N > 0$, λ is in the stability region if (A_δ, H_δ) is detectable. Thus, the detectability of (A_δ, H_δ) guarantees the detectability of (F_e, H_e), for integers $N > 0$. □

B.3 Derivation of Decentralized Controller for Sampled-data Two-rate Problem

The derivation of the decentralized controller design in Theorem 3.3.3 is presented next.

Theorem For the discrete-time decentralized plant

$$\delta x = A_\delta x + \sum_{i=1}^{p} B_\delta^i u_i + (\sqrt{N} G_\delta) \tilde{w}, \quad x(0) = 0,$$

with the performance-output variable

$$\tilde{z} = \begin{pmatrix} \frac{1}{\sqrt{N}} \left(H_\delta x + \sum_{j=1}^{p} D_{BH}^j u_j\right) + D_{GH} \tilde{w} \\ u \end{pmatrix},$$

with (A_δ, H_δ) detectable and with observer-based decentralized discrete-time controller $u_i = K_i^c \xi_i$, where

$$\delta\xi_i = \left(A_\delta + B_\delta^i K_i^c + \sum_{j\neq i} B_\delta^j K_j^c + (\sqrt{N} G_\delta) K^d\right) \xi_i + K_i^o(y_i - C_i\xi_i), \quad \xi_i(0) = 0,$$

a sufficient condition to guarantee that the closed-loop system is stable and has H_∞ norm from $\binom{\tilde{w}}{w}$ to $\tilde{z}$ less than a given value α is

$$\begin{aligned}
K^c = -[&I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\
&+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
\cdot [&B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\
&+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} T G_\delta' X (I + TA_\delta)\right)],
\end{aligned}$$

$$K^d = \Upsilon^{-1}\left(\frac{1}{\sqrt{N}} D_{GH}'(H_\delta + D_{BH}K^c) + \sqrt{N} G_\delta' X(I + T(A_\delta + B_\delta K^c))\right),$$

where

$$\Upsilon := \alpha^2 I - TNG_\delta' X G_\delta - D_{GH}' D_{GH},$$

and K_D^o block diagonal, $X > 0$, and $W > 0$ satisfy

$$\begin{aligned}
0 = {}& A_\delta' X + X A_\delta + T A_\delta' X A_\delta + \tfrac{1}{N} H_\delta' H_\delta \\
&+ \left(\tfrac{1}{\sqrt{N}} H_\delta' D_{GH} + \sqrt{N}(I + TA_\delta)' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right) \\
&- [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]' \\
&\cdot [I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
&\cdot [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]
\end{aligned}$$

and

$$\begin{aligned}
W = {}& TNG_{\delta,C} \Upsilon^{-1} G_{\delta,C}' + T \tfrac{1}{\alpha^2} K_D^o K_D^{o\prime} \\
&+ [I + T(A_e - G_{\delta,C} \Upsilon^{-1} D_{GH}' D_{BH} K_D^c - TNG_{\delta,C} \Upsilon^{-1} G_\delta' X B_\delta K_D^c) - TK_D^o C_D] \\
&\quad \cdot (W^{-1} - TK_D^{c\prime}[I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\
&\quad\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)] K_D^c)^{-1} \\
&\quad \cdot [I + T(A_e - G_{\delta,C} \Upsilon^{-1} D_{GH}' D_{BH} K_D^c - TNG_{\delta,C} \Upsilon^{-1} G_\delta' X B_\delta K_D^c) - TK_D^o C_D]',
\end{aligned}$$

such that the eigenvalues of $A_e - K_D^o C_D$ are in the stability region and

$$X^{-1} - TNG_\delta(\alpha^2 I - D'_{GH}D_{GH})^{-1}G'_\delta > 0,$$

$$W^{-1} - TK_D^{c\prime}[I + TB'_\delta X B_\delta + \tfrac{1}{N}D'_{BH}D_{BH} + (\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta X G_\delta)\Upsilon^{-1}(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_\delta X B_\delta)]K_D^c > 0,$$

and

$$|I + T(A_e - G_{\delta,C}\Upsilon^{-1}D'_{GH}D_{BH}K_D^c - TNG_{\delta,C}\Upsilon^{-1}G'_\delta X B_\delta K_D^c) - TK_D^o C_D| \neq 0.$$

Proof Now consider Problem 3.2.4. Given the form chosen for the controller and observers, we find the closed-loop system matrices F_e, G_e, H_e, and E_e. Let $e_i := \xi_i - x$, $i = 1, \ldots, p$. Then,

$$\delta \begin{pmatrix} x \\ e_1 \\ \vdots \\ e_p \end{pmatrix} = F_e \begin{pmatrix} x \\ e_1 \\ \vdots \\ e_p \end{pmatrix} + G_e \begin{pmatrix} \tilde{w} \\ w_1 \\ \vdots \\ w_p \end{pmatrix}$$

$$\tilde{z} = H_e \begin{pmatrix} x \\ e_1 \\ \vdots \\ e_p \end{pmatrix} + E_e \begin{pmatrix} \tilde{w} \\ w_1 \\ \vdots \\ w_p \end{pmatrix}$$

where

$$F_e = \begin{pmatrix} A_\delta + \sum_{i=1}^p B_\delta^j K_j^c & B_\delta^1 K_1^c & \cdots & B_\delta^p K_p^c \\ \sqrt{N}G_\delta K^d & (A_\delta + \sum_{j\neq 1} B_\delta^j K_j^c + \sqrt{N}G_\delta K^d) - K_1^o C_1 & & -B_\delta^p K_p^c \\ \vdots & & \ddots & \\ \sqrt{N}G_\delta K^d & -B_\delta^1 K_1^c & & (A_\delta + \sum_{j\neq p} B_\delta^j K_j^c + \sqrt{N}G_\delta K^d) - K_p^o C_p \end{pmatrix},$$

$$H_e = \begin{pmatrix} \frac{1}{\sqrt{N}}\left(H_\delta + \sum_{j=1}^p D_{BH}^j K_j^c\right) & \frac{1}{\sqrt{N}}D_{BH}^1 K_1^c & \cdots & \frac{1}{\sqrt{N}}D_{BH}^p K_p^c \\ K_1^c & K_1^c & & 0 \\ \vdots & & \ddots & \\ K_p^c & 0 & & K_p^c \end{pmatrix},$$

and

$$G_e = \begin{pmatrix} \sqrt{N}G_\delta & 0 & \cdots & 0 \\ -\sqrt{N}G_\delta & K_1^o & & 0 \\ \vdots & & \ddots & \\ -\sqrt{N}G_\delta & 0 & & K_p^o \end{pmatrix}, \qquad E_e = \begin{pmatrix} D_{GH} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

This can be written in a shorthand form if the following composite matrices are defined:

$$e := \begin{pmatrix} e_1 \\ \vdots \\ e_p \end{pmatrix}, \quad w := \begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix}, \quad K^c := \begin{pmatrix} K_1^c \\ \vdots \\ K_p^c \end{pmatrix}, \quad G_{\delta,C} := \begin{pmatrix} G_\delta \\ \vdots \\ G_\delta \end{pmatrix},$$

$$C_D := \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & C_p \end{pmatrix}, \quad K_D^c := \begin{pmatrix} K_1^c & 0 & \cdots & 0 \\ 0 & K_2^c & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K_p^c \end{pmatrix}, \quad K_D^o := \begin{pmatrix} K_1^o & 0 & \cdots & 0 \\ 0 & K_2^o & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K_p^o \end{pmatrix},$$

$$B_\delta := (B_\delta^1 \; B_\delta^2 \cdots B_\delta^p), \qquad D_{BH} := (D_{BH}^1 \; D_{BH}^2 \cdots D_{BH}^p),$$

and

$$A_e := \begin{pmatrix} A_\delta + B_\delta K^c + \sqrt{N} G_\delta K^d & 0 & \cdots & & 0 \\ 0 & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & \cdots & 0 & A_\delta + B_\delta K^c + \sqrt{N} G_\delta K^d \end{pmatrix}$$

$$+ \begin{pmatrix} -B_\delta^1 K_1^c & -B_\delta^2 K_2^c & \cdots & -B_\delta^p K_p^c \\ -B_\delta^1 K_1^c & \ddots & & \vdots \\ \vdots & & \ddots & -B_\delta^p K_p^c \\ -B_\delta^1 K_1^c & \cdots & -B_\delta^{p-1} K_{p-1}^c & -B_\delta^p K_p^c \end{pmatrix}.$$

The shorthand for the closed-loop system is

$$\delta \begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} A_\delta + B_\delta K^c & B_\delta K_D^c \\ \sqrt{N} G_{\delta,C} K^d & A_e - K_D^o C_D \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} \sqrt{N} G_\delta & 0 \\ -\sqrt{N} G_{\delta,C} & K_D^o \end{pmatrix} \begin{pmatrix} \tilde{w} \\ w \end{pmatrix},$$

$$\tilde{z} = \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta + D_{BH} K^c) & \frac{1}{\sqrt{N}} D_{BH} K_D^c \\ K^c & K_D^c \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} D_{GH} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{w} \\ w \end{pmatrix}.$$

Choosing $X_e \geq 0$ to have the diagonal form $\left(\begin{smallmatrix} X & 0 \\ 0 & X_1 \end{smallmatrix}\right) \geq 0$ where $X \geq 0$ and $X_1 > 0$, and expanding condition (i) of the bounded real lemma, we find conditions on K^c, K^o, and K^d. The upper-left element becomes

$$
\begin{aligned}
0 = {} & (A_\delta+B_\delta K^c)'X+X(A_\delta+B_\delta K^c)+T(A_\delta+B_\delta K^c)'X(A_\delta+B_\delta K^c)+K^{c\prime}K^c \\
& +\tfrac{1}{N}(H_\delta+D_{BH}K^c)'(H_\delta+D_{BH}K^c) \\
& +\left(\tfrac{1}{\sqrt{N}}(H_\delta+D_{BH}K^c)'D_{GH}+\sqrt{N}(I+T(A_\delta+B_\delta K^c))'XG_\delta\right) \\
& \quad \cdot \Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}(H_\delta+D_{BH}K^c)+\sqrt{N}G'_\delta X(I+T(A_\delta+B_\delta K^c))\right) \\
& +TN\left(\tfrac{1}{\sqrt{N}}(H_\delta+D_{BH}K^c)'D_{GH}\Upsilon^{-1}G'_{\delta,C}+\sqrt{N}(I+T(A_\delta+B_\delta K^c))'XG_\delta\Upsilon^{-1}G'_{\delta,C}-K^{d\prime}G'_{\delta,C}\right) \\
& \quad \cdot X_1 D^{-1} \\
& \quad \cdot\left(\tfrac{1}{\sqrt{N}}G_{\delta,C}\Upsilon^{-1}D'_{GH}(H_\delta+D_{BH}K^c)+\sqrt{N}G_{\delta,C}\Upsilon^{-1}G'_\delta X(I+T(A_\delta+B_\delta K^c))-G_{\delta,C}K^d\right),
\end{aligned}
$$

where

$$\Upsilon := \alpha^2 I - TNG'_\delta XG_\delta - D'_{GH}D_{GH}$$

and

$$D := I - TNG_{\delta,C}\Upsilon^{-1}G'_{\delta,C}X_1 - T\frac{1}{\alpha^2}K^o_D K^{o\prime}_D X_1.$$

The final term will be zero if K^d is chosen to satisfy

$$K^d = \Upsilon^{-1}\left(\frac{1}{\sqrt{N}}D'_{GH}(H_\delta + D_{BH}K^c) + \sqrt{N}G'_\delta X(I + T(A_\delta + B_\delta K^c))\right).$$

The remaining equation that must be satisfied is then

$$
\begin{aligned}
0 = {} & (A_\delta + B_\delta K^c)'X + X(A_\delta + B_\delta K^c) + T(A_\delta + B_\delta K^c)'X(A_\delta + B_\delta K^c) \\
& + K^{c\prime}K^c + \tfrac{1}{N}(H_\delta + D_{BH}K^c)'(H_\delta + D_{BH}K^c) \\
& + \left(\tfrac{1}{\sqrt{N}}(H_\delta + D_{BH}K^c)'D_{GH} + \sqrt{N}(I + T(A_\delta + B_\delta K^c))'XG_\delta\right)\Upsilon^{-1} \\
& \quad \cdot\left(\tfrac{1}{\sqrt{N}}D'_{GH}(H_\delta + D_{BH}K^c) + \sqrt{N}G'_\delta X(I + T(A_\delta + B_\delta K^c))\right).
\end{aligned} \tag{B.1}
$$

Given the above choice of K^d, the upper-right element in condition (i) of the bounded real lemma reduces to

$$
\begin{aligned}
0 = {} & K^{c\prime}[I + TB'_\delta XB_\delta + \tfrac{1}{N}D'_{BH}D_{BH} \\
& \quad + \left(\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_\delta XB_\delta\right)]K^c_D \\
& + [(I + TA_\delta)'XB_\delta + \tfrac{1}{N}H'_\delta D_{BH} \\
& \quad + \left(\tfrac{1}{\sqrt{N}}H'_\delta D_{GH} + \sqrt{N}(I + TA_\delta)'XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_\delta XB_\delta\right)]K^c_D,
\end{aligned}
$$

which is satisfied by

$$K^c = [I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH}$$
$$+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1}$$
$$\cdot [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta$$
$$+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} T G_\delta' X (I + TA_\delta)\right)].$$

This is the same form as the controller gain equation in the state-feedback case. In fact, with this choice of K^c, (B.1) reduces to the same design equation as in the state-feedback case.

The lower-right element of the bounded real lemma reduces, for $T \neq 0$, to

$$X_1 = [I + T(A_e' - K_D^{c\prime} D_{BH}' D_{GH} \Upsilon^{-1} G_{\delta,C}' - TN K_D^{c\prime} B_\delta' X G_\delta \Upsilon^{-1} G_{\delta,C}') - TC_D' K_D^{o\prime}] X_1 D^{-1}$$
$$\cdot [I + T(A_e - G_{\delta,C} \Upsilon^{-1} D_{GH}' D_{BH} K_D^c - TN G_{\delta,C} \Upsilon^{-1} G_\delta' X B_\delta K_D^c) - TK_D^o C_D]$$
$$+ TK_D^{c\prime} [I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH}$$
$$+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)] K_D^c.$$

Subtracting the constant term and multiplying on either side by

$$[I + T(A_e - G_{\delta,C} \Upsilon^{-1} D_{GH}' D_{BH} K_D^c - TN G_{\delta,C} \Upsilon^{-1} G_\delta' X B_\delta K_D^c) - TK_D^o C_D]^{-1},$$

or its transpose, yield

$$(X_1^{-1} - TN G_{\delta,C} \Upsilon^{-1} G_{\delta,C}' - T \tfrac{1}{\alpha^2} K_D^o K_D^{o\prime})^{-1}$$
$$= [I + T(A_e - G_{\delta,C} \Upsilon^{-1} D_{GH}' D_{BH} K_D^c - TN G_{\delta,C} \Upsilon^{-1} G_\delta' X B_\delta K_D^c) - TK_D^o C_D]'^{-1}$$
$$\cdot (X_1 - TK_D^{c\prime} [I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH}$$
$$+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)] K_D^c)$$
$$\cdot [I + T(A_e - G_{\delta,C} \Upsilon^{-1} D_{GH}' D_{BH} K_D^c - TN G_{\delta,C} \Upsilon^{-1} G_\delta' X B_\delta K_D^c) - TK_D^o C_D]^{-1}.$$

Inverting and subtracting the constant term, then substituting $W := X_1^{-1}$, we obtain

$$W = TN G_{\delta,C} \Upsilon^{-1} G_{\delta,C}' + T \tfrac{1}{\alpha^2} K_D^o K_D^{o\prime}$$
$$+ [I + T(A_e - G_{\delta,C} \Upsilon^{-1} D_{GH}' D_{BH} K_D^c - TN G_{\delta,C} \Upsilon^{-1} G_\delta' X B_\delta K_D^c) - TK_D^o C_D]$$
$$\cdot (W^{-1} - TK_D^{c\prime} [I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH}$$
$$+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)] K_D^c)^{-1}$$
$$\cdot [I + T(A_e - G_{\delta,C} \Upsilon^{-1} D_{GH}' D_{BH} K_D^c - TN G_{\delta,C} \Upsilon^{-1} G_\delta' X B_\delta K_D^c) - TK_D^o C_D]'.$$

Condition (ii) of the bounded real lemma is simply

$$\begin{pmatrix} \alpha^2 I - TN G_\delta' X G_\delta - D_{GH}' D_{GH} - TN G_{\delta,C}' X_1 G_{\delta,C} & T\sqrt{N} G_{\delta,C}' X_1 K_D^o \\ T\sqrt{N} K_D^{o\prime} X_1 G_{\delta,C} & \alpha^2 I - TK_D^{o\prime} X_1 K_D^o \end{pmatrix} > 0.$$

However, as N increases, so does the dimension of this matrix. Thus, we require equivalent conditions that do not have that property. If $X > 0$, then X_e^{-1} exists. We can show that

$$\alpha^2 I - E_e' E_e - T G_e' X_e G_e > 0$$

is equivalent to

$$X_e^{-1} - T G_e(\alpha^2 I - E_e' E_e)^{-1} G_e' > 0,$$

as long as $\alpha^2 I - E_e' E_e > 0$. This is satisfied if $\alpha^2 I - D_{GH}' D_{GH} > 0$. Using the matrix identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}, \qquad \text{if } |A| \neq 0,$$

we show that $X_e^{-1} - T G_e(\alpha^2 I - E_e' E_e)^{-1} G_e' > 0$ holds if both

$$X^{-1} - TN G_\delta(\alpha^2 I - D_{GH}' D_{GH})^{-1} G_\delta' > 0$$

and

$$W - TN G_{\delta,C}(\alpha^2 I - TN G_\delta' X G_\delta - D_{GH}' D_{GH})^{-1} G_{\delta,C}' - \frac{1}{\alpha^2} T K_D^o K_D^{o\prime} > 0.$$

The first of these conditions is equivalent to $\alpha^2 I - TN G_\delta' X G_\delta - D_{GH}' D_{GH} > 0$, which implies $\alpha^2 I - D_{GH}' D_{GH} > 0$. The second condition is equivalent to

$$\begin{aligned} W^{-1} - T K_D^{c\prime}[I + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ + (\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta)\Upsilon^{-1}(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta)] K_D^c > 0, \end{aligned}$$

as long as

$$|I + T(A_e - G_{\delta,C}\Upsilon^{-1} D_{GH}' D_{BH} K_D^c - TN G_{\delta,C}\Upsilon^{-1} G_\delta' X B_\delta K_D^c) - T K_D^o C_D| \neq 0.$$

Now consider under what conditions (F_e, H_e) is detectable. Let λ, $\binom{v_1}{v_2}$, be an eigenvalue-eigenvector pair for F_e, H_e, such that

$$\begin{pmatrix} A_\delta + B_\delta K^c & B_\delta K_D^c \\ \sqrt{N} G_{\delta,C} K^d & A_e - K_D^o C_D \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta + D_{BH} K^c) & \frac{1}{\sqrt{N}} D_{BH} K_D^c \\ K^c & K_D^c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$$

Thus, $A_\delta v_1 = \lambda v_1$, $\frac{1}{\sqrt{N}} H_\delta v_1 = 0$. For all integers $N > 0$, either $v_1 = 0$ or λ is in the stability region if (A_δ, H_δ) is detectable. Suppose $v_1 = 0$. Then $(A_e - K_D^o C_D) v_2 = \lambda v_2$. Thus, if $A_e - K_D^o C_D$ has eigenvalues in the stability region, then λ is also in the stability region. $\square$

B.4 Derivation of Reliable Decentralized Controller for Sampled-data Two-rate Problem

Next, the derivation of the reliable controller design of Theorem 3.5.1 is presented.

Theorem For the decentralized system (3.10), (3.11), (3.12), with (A_δ, H_δ) detectable and with observer-based feedback $u_i = K_i^c \xi_i$, with observer (3.13), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{\tilde{w}_e \tilde{z}}\|_\infty \leq \alpha$ for all subsets of subsystem sensor failures $\omega \subseteq \Omega$ is

$$\begin{aligned} K^c = &-[I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ &\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\ &\cdot [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\ &\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} T G_\delta' X (I + TA_\delta)\right)], \end{aligned}$$

$$K^d = \Upsilon^{-1} \left(\frac{1}{\sqrt{N}} D_{GH}' (H_\delta + D_{BH} K^c) + \sqrt{N} G_\delta' X (I + T(A_\delta + B_\delta K^c)) \right),$$

where

$$\Upsilon := \alpha^2 I - TNG_\delta' X G_\delta - D_{GH}' D_{GH},$$

and K_D^o block diagonal, $X > 0$, and $W > 0$ satisfy

$$\begin{aligned} 0 = &A_\delta' X + X A_\delta + T A_\delta' X A_\delta + \tfrac{1}{N} H_\delta' H_\delta + \alpha^2 C_{\delta,\Omega}' C_{\delta,\Omega} \\ &\quad + \left(\tfrac{1}{\sqrt{N}} H_\delta' D_{GH} + \sqrt{N} (I + TA_\delta)' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right) \\ &- [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\ &\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)]' \\ &\cdot [k_D' k_D + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ &\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\ &\cdot [B_\delta' X (I + TA_\delta) + \tfrac{1}{N} D_{BH}' H_\delta \\ &\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + TA_\delta)\right)] \end{aligned}$$

and

$$\begin{aligned} W = &TNG_{\delta,C} \Upsilon^{-1} G_{\delta,C}' + T \tfrac{1}{\alpha^2} K_D^o K_D^{o\prime} \\ &+ [I + T(A_e - G_{\delta,C} \Upsilon^{-1} [D_{GH}' D_{BH} + TNG_\delta' X B_\delta] K_D^c) - TK_D^o C_D] \\ &\cdot (W^{-1} - TK_D^{o\prime} [I + TB_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} \\ &\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)] K_D^c)^{-1} \\ &\cdot [I + T(A_e - G_{\delta,C} \Upsilon^{-1} [D_{GH}' D_{BH} + TNG_\delta' X B_\delta] K_D^c) - TK_D^o C_D]', \end{aligned}$$

such that the eigenvalues of $A_e - K^o_D C_D$ are in the stability region,

$$X^{-1} - TNG_\delta(\alpha^2 I - D'_{GH}D_{GH})^{-1}G'_\delta > 0,$$

$$W^{-1} - TK^{c\prime}_D[I + TB'_\delta X B_\delta + \tfrac{1}{N}D'_{BH}D_{BH}$$
$$+ (\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta)\Upsilon^{-1}(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_\delta XB_\delta)]K^c_D > 0,$$

and

$$|I + T(A_e - G_{\delta,C}\Upsilon^{-1}[D'_{GH}D_{BH} + TNG'_\delta XB_\delta]K^c_D) - TK^o_D C_D| \neq 0.$$

Proof Suppose that the sensors with indices in the set $\omega \subseteq \Omega$ experience outages $y_i = 0$, $\forall i \in \omega$. Then the system matrices change to

$$\delta\begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} A_\delta + B_\delta K^c & B_\delta K^c_D \\ \sqrt{N}G_{\delta,C}K^d - K^o_{D,\omega}C_\omega & A_e - K^o_D C_D \end{pmatrix}\begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} \sqrt{N}G_\delta & 0 \\ -\sqrt{N}G_{\delta,C} & K^o_{D,\bar{\omega}} \end{pmatrix}\begin{pmatrix} \tilde{w} \\ w \end{pmatrix},$$

$$\tilde{z} = \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta + D_{BH}K^c) & \frac{1}{\sqrt{N}}D_{BH}K^c_D \\ K^c & K^c_D \end{pmatrix}\begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} D_{GH} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \tilde{w} \\ w \end{pmatrix},$$

where

- $K^o_{D,\omega}$ is K^o_D with diagonal blocks not in ω set equal to zero
- C_ω is C with the blocks not in ω set equal to zero
- $K^o_{D,\bar{\omega}} := K^o_D - K^o_{D,\omega}$.

Let these new system matrices be denoted $\bar{F}_e$, $\bar{G}_e$, $\bar{H}_e$, and $\bar{E}_e$. Then,

$$\bar{F}_e = F_e - K^o_{\omega e}C_{\omega e}, \quad \bar{G}_e = G_e - K^o_{\omega e}(0 \;\; I), \quad \bar{H}_e = H_e, \quad \bar{E}_e = E_e,$$

where

$$K^o_{\omega e} := \begin{pmatrix} 0 \\ K^o_{D,\omega} \end{pmatrix}, \qquad C_{\omega e} := (C_\omega \;\; 0).$$

Define also $C_{\Omega e} := (C_\Omega \;\; 0)$.

Now, condition (i) of the generalized bounded real lemma may be rewritten in terms of the original system matrices. Note that

$$\left(\alpha^2 I - E'_e E_e\right)^{-1} = \begin{pmatrix} (\alpha^2 I - D'_{GH}D_{GH})^{-1} & 0 \\ 0 & \frac{1}{\alpha^2}I \end{pmatrix}$$

and

$$E_e \begin{pmatrix} 0 \\ I \end{pmatrix} = 0.$$

Then, the terms in the expression can be reduced as follows:

$$\bar{G}_e(\alpha^2 I - E_e'E_e)^{-1}\bar{G}_e' = G_e(\alpha^2 I - E_e'E_e)^{-1}G_e' - \frac{1}{\alpha^2}K_{we}^o K_{we}^{o\prime}.$$

Thus,

$$\begin{aligned}
&(\alpha^2 I - T\bar{G}_e'X_e\bar{G}_e - \bar{E}_e'\bar{E}_e)^{-1} \\
&\quad = T\left((\alpha^2 I - E_e'E_e)^{-1}G_e' - \tfrac{1}{\alpha^2}\left(\begin{smallmatrix}0\\I\end{smallmatrix}\right)K_{we}^{o\prime}\right) \\
&\qquad \cdot X_e\left(I - TG_e(\alpha^2 I - E_e'E_e)^{-1}G_e'X_e + T\tfrac{1}{\alpha^2}K_{we}^o K_{we}^{o\prime}X_e\right)^{-1} \\
&\qquad \cdot\left(G_e(\alpha^2 I - E_e'E_e)^{-1} - \tfrac{1}{\alpha^2}K_{we}^o(0\ \ I)\right) + (\alpha^2 I - E_e'E_e)^{-1},
\end{aligned}$$

and

$$\begin{aligned}
&(\bar{H}_e'\bar{E}_e + (I + T\bar{F}_e)'X_e\bar{G}_e)(\alpha^2 I - T\bar{G}_e'X_e\bar{G}_e - \bar{E}_e'\bar{E}_e)^{-1}(\bar{E}_e'\bar{H}_e + \bar{G}_e'X_e(I + T\bar{F}_e)) \\
&\quad = (H_e'E_e + (I + T\bar{F}_e)'X_e(G_e - K_{we}^o(0\ \ I)))\,(\alpha^2 I - E_e'E_e)^{-1} \\
&\qquad \cdot\left(\left(G_e' - \left(\begin{smallmatrix}0\\I\end{smallmatrix}\right)K_{we}^{o\prime}\right)X_e(I + T\bar{F}_e) + E_e'H_e\right) \\
&\quad + \left(H_e'E_e(\alpha^2 I - E_e'E_e)^{-1}G_e'X_e + (I + T\bar{F}_e)'X_e\left(G_e(\alpha^2 I - E_e'E_e)^{-1}G_e'X_e - \tfrac{1}{\alpha^2}K_{we}^o K_{we}^{o\prime}X_e\right)\right) \\
&\qquad \cdot\left(I - TG_e(\alpha^2 I - E_e'E_e)^{-1}G_e'X_e + T\tfrac{1}{\alpha^2}K_{we}^o K_{we}^{o\prime}X_e\right)^{-1} \\
&\qquad \cdot\left(G_e(\alpha^2 I - E_e'E_e)^{-1}E_e'H_e + \left(G_e(\alpha^2 I - E_e'E_e)^{-1}G_e'X_e - \tfrac{1}{\alpha^2}K_{we}^o K_{we}^{o\prime}X_e\right)(I + T\bar{F}_e)\right).
\end{aligned}$$

Expanding condition (i) of the generalized bounded real lemma in this manner, and regrouping terms, we find that

$$\begin{aligned}
&\bar{F}_e'X_e + X_e\bar{F}_e + T\bar{F}_e'X_e\bar{F}_e + \bar{H}_e'\bar{H}_e \\
&\qquad + (\bar{H}_e'\bar{E}_e + (I + T\bar{F}_e)'X_e\bar{G}_e)(\alpha^2 I - T\bar{G}_e'X_e\bar{G}_e - \bar{E}_e'\bar{E}_e)^{-1}(\bar{E}_e'\bar{H}_e + \bar{G}_e'X_e(I + T\bar{F}_e)) \\
&\quad = F_e'X_e + X_eF_e + TF_e'X_eF_e + H_e'H_e + \alpha^2 C_{we}'C_{we} \\
&\qquad + (H_e'E_e + (I + TF_e)'X_eG_e)(\alpha^2 I - TG_e'X_eG_e - E_e'E_e)^{-1}(E_e'H_e + G_e'X_e(I + TF_e)) \\
&\quad - \left(\tfrac{1}{\alpha}(I + TF_e)'X_eL_e^{-1}K_{we}^o + \alpha C_{we}' + \tfrac{1}{\alpha}TH_e'E_e(\alpha^2 I - E_e'E_e)^{-1}G_e'X_eL_e^{-1}K_{we}^o\right) \\
&\qquad \cdot\left(I + T\tfrac{1}{\alpha^2}K_{we}^{o\prime}X_eL_e^{-1}K_{we}^o\right)^{-1} \\
&\qquad \cdot\left(\tfrac{1}{\alpha}K_{we}^{o\prime}X_eL_e^{-1}(I + TF_e) + \alpha C_{we} + \tfrac{1}{\alpha}TK_{we}^{o\prime}X_eL_e^{-1}G_e(\alpha^2 I - E_e'E_e)^{-1}E_e'H_e\right)
\end{aligned}$$

where

$$L_e := I - TG_e(\alpha^2 I - E_e'E_e)^{-1}G_e'X_e.$$

Thus,

$$\begin{aligned}
&\bar{F}_e'X_e + X_e\bar{F}_e + T\bar{F}_e'X_e\bar{F}_e + \bar{H}_e'\bar{H}_e \\
&\qquad + (\bar{H}_e'\bar{E}_e + (I + T\bar{F}_e)'X_e\bar{G}_e)(\alpha^2 I - T\bar{G}_e'X_e\bar{G}_e - \bar{E}_e'\bar{E}_e)^{-1}(\bar{E}_e'\bar{H}_e + \bar{G}_e'X_e(I + T\bar{F}_e)) \le 0
\end{aligned}$$

for all $\omega \subseteq \Omega$ if

$$F_e'X_e + X_eF_e + TF_e'X_eF_e + H_e'H_e + \alpha^2 C_{\Omega e}'C_{\Omega e} + (H_e'E_e + (I+TF_e)'X_eG_e)(\alpha^2 I - TG_e'X_eG_e - E_e'E_e)^{-1}(E_e'H_e + G_e'X_e(I+TF_e)) \leq 0 \tag{B.2}$$

and

$$I + T\frac{1}{\alpha^2}K_{\omega e}^{o\prime}X_eL_e{}^{-1}K_{\omega e}^{o} > 0.$$

Expanding $\alpha^2 C_{\Omega e}'C_{\Omega e}$,

$$\alpha^2 C_{\Omega e}'C_{\Omega e} = \begin{pmatrix} \alpha^2 C_\Omega' C_\Omega & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, when expanding out (B.2), the extra term $\alpha^2 C_\Omega' C_\Omega$ enters at the upper-left element. This results in the extra term in the design equation for X, (3.26), and no changes from the basic design in the expressions for K^c, K^d, and the design equation for W.

Condition (ii) of the generalized bounded real lemma and $I + T\frac{1}{\alpha^2}K_{\omega e}^{o\prime}X_eL_e{}^{-1}K_{\omega e}^{o} > 0$ are satisfied for all $\omega \subseteq \Omega$ if $X > 0$, $X^{-1} - TNG_\delta(\alpha^2 I - D_{GH}'D_{GH})^{-1}G_\delta' > 0$, and

$$W - TNG_{\delta,C}(\alpha^2 I - D_{GH}'D_{GH} - TNG_\delta'XG_\delta)^{-1}G_{\delta,C}' - T\frac{1}{\alpha^2}K_D^oK_D^{o\prime} > 0.$$

(This last condition implies that

$$W - TNG_{\delta,C}(\alpha^2 I - D_{GH}'D_{GH} - TNG_\delta'XG_\delta)^{-1}G_{\delta,C}' - T\frac{1}{\alpha^2}K_D^oK_D^{o\prime} + T\frac{1}{\alpha^2}K_{D,\omega}^oK_{D,\omega}^{o\prime} > 0,$$

which is required by condition (ii).) The condition on W is equivalent to

$$W^{-1} - TK_D^{o\prime}[I + TB_\delta'XB_\delta + \tfrac{1}{N}D_{BH}'D_{BH} + (\tfrac{1}{\sqrt{N}}D_{BH}'D_{GH} + \sqrt{N}TB_\delta'XG_\delta)\Upsilon^{-1}(\tfrac{1}{\sqrt{N}}D_{GH}'D_{BH} + \sqrt{N}TG_\delta'XB_\delta)]K_D^o > 0,$$

if

$$|I + T(A_e - G_{\delta,C}\Upsilon^{-1}[D_{GH}'D_{BH} - TNG_\delta'XB_\delta]K_D^c) - TK_D^oC_D| \neq 0.$$

Finally, the same detectability proof follows as before since only the lower-left element of F_e is affected by the outages and it has no effect on the proof. □

APPENDIX C

PROOFS OF THEOREMS IN CHAPTER 4

C.1 Derivation of Reliable Decentralized Controller Design for Multirate Problem

The following is the derivation of the sensor-outage reliable decentralized controller design for the multirate controller design problem in Theorem 4.4.2.

Theorem For the decentralized system (4.11), (4.12), (4.13), with (A_δ, H_δ) detectable and with observer-based feedback $u_i = K_i^c \xi_i$, with observer (4.14), a sufficient condition to guarantee that the closed-loop plant is stable and that $\|T_{\binom{\tilde{w}^0}{v}\tilde{z}}\|_\infty \leq \alpha$ for all subsets of subsystem sensor failures $\omega \subseteq \Omega$ is

$$
\begin{aligned}
K^c = -[&k_D' k_D + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} + \tfrac{1}{N}\alpha^2 D_{BC,\Omega}' D_{BC,\Omega} \\
&+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
\cdot [&B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \tfrac{1}{N}\alpha^2 D_{BC,\Omega}' C_{\delta,\Omega} \\
&+ \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right)],
\end{aligned}
$$

$$
K^d = \Upsilon^{-1} \left(\frac{1}{\sqrt{N}} D_{GH}' (H_\delta + D_{BH} K^c) + \sqrt{N} G_\delta' X (I + T(A_\delta + B_\delta K^c))\right),
$$

where

$$
\Upsilon := \alpha^2 I - T N G_\delta' X G_\delta - D_{GH}' D_{GH},
$$

and K_D^o block diagonal, $X > 0$, and $W > 0$ satisfy

$$
\begin{aligned}
0 = {} & A_\delta' X + X A_\delta + T A_\delta' X A_\delta + \tfrac{1}{N} H_\delta' H_\delta + \tfrac{1}{N}\alpha^2 C_{\delta,\Omega}' C_{\delta,\Omega} \\
&+ \left(\tfrac{1}{\sqrt{N}} H_\delta' D_{GH} + \sqrt{N} (I + T A_\delta)' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right) \\
&- [B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \tfrac{1}{N}\alpha^2 D_{BC,\Omega}' C_{\delta,\Omega} \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right)]' \\
&\cdot [k_D' k_D + T B_\delta' X B_\delta + \tfrac{1}{N} D_{BH}' D_{BH} + \tfrac{1}{N}\alpha^2 D_{BC,\Omega}' D_{BC,\Omega} \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' D_{BH} + \sqrt{N} T G_\delta' X B_\delta\right)]^{-1} \\
&\cdot [B_\delta' X (I + T A_\delta) + \tfrac{1}{N} D_{BH}' H_\delta + \tfrac{1}{N}\alpha^2 D_{BC,\Omega}' C_{\delta,\Omega} \\
&\quad + \left(\tfrac{1}{\sqrt{N}} D_{BH}' D_{GH} + \sqrt{N} T B_\delta' X G_\delta\right) \Upsilon^{-1} \left(\tfrac{1}{\sqrt{N}} D_{GH}' H_\delta + \sqrt{N} G_\delta' X (I + T A_\delta)\right)]
\end{aligned}
$$

and

$$W = TNG_{\delta,C}\Upsilon^{-1}G'_{\delta,C} + TN\tfrac{1}{\alpha^2}K^o_D K^{o\prime}_D$$
$$+ [I + T(A_e - G_{\delta,C}\Upsilon^{-1}[D'_{GH}D_{BH} + TNG'_\delta XB_\delta]K^c_D)$$
$$- TK^o_D(C_D + D_{BC,D}K^c_{C,D} - D_{BC}K^c_D)]$$
$$\cdot (W^{-1} - TK^{o\prime}_D[k'_D k_D + TB'_\delta XB_\delta + \tfrac{1}{N}D'_{BH}D_{BH} + \tfrac{1}{N}\alpha^2 D'_{BC,\Omega}D_{BC,\Omega}$$
$$+ \left(\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_\delta XB_\delta\right)]K^c_D)^{-1}$$
$$\cdot [I + T(A_e - G_{\delta,C}\Upsilon^{-1}[D'_{GH}D_{BH} + TNG'_\delta XB_\delta]K^c_D)$$
$$- TK^o_D(C_D + D_{BC,D}K^c_{C,D} - D_{BC}K^c_D)]',$$

such that the eigenvalues of $A_e - K^o_D(C_D + D_{BC,D}K^c_{C,D})$ are in the stability region, $\Upsilon > 0$,

$$W^{-1} - TK^{o\prime}_D[k'_D k_D + TB'_\delta XB_\delta + \tfrac{1}{N}D'_{BH}D_{BH} + \tfrac{1}{N}\alpha^2 D'_{BC,\Omega}D_{BC,\Omega}$$
$$+ (\tfrac{1}{\sqrt{N}}D'_{BH}D_{GH} + \sqrt{N}TB'_\delta XG_\delta)\Upsilon^{-1}(\tfrac{1}{\sqrt{N}}D'_{GH}D_{BH} + \sqrt{N}TG'_\delta XB_\delta)]K^c_D > 0,$$

and

$$|I + T(A_e - G_{\delta,C}\Upsilon^{-1}[D'_{GH}D_{BH} + TNG'_\delta XB_\delta]K^c_D)$$
$$- TK^o_D(C_D + D_{BC,D}K^c_{C,D} - D_{BC}K^c_D)| \neq 0.$$

Proof This proof differs from other sensor-outage reliable proofs in the way in which the terms in $\alpha^2 C'_{\Omega e}C_{\Omega e}$ modify the design equations, as will be seen. As before, the system matrices with and without outages are first found and the required changes to the design equations are found to meet the conditions of the generalized bounded real lemma for all possible outages in the prespecified set.

The closed-loop system matrices when no outages occur are

$$F_e = \begin{pmatrix} A_\delta + B_\delta K^c & B_\delta K^c_D \\ \sqrt{N}G_{\delta,C}K^d & A_e - K^o_D(C_D + D_{BC,D}K^c_{C,D} - D_{BC}K^c_D) \end{pmatrix},$$

$$G_e = \begin{pmatrix} \sqrt{N}G_\delta & 0 \\ -\sqrt{N}G_{\delta,C} & \sqrt{N}K^o_D \end{pmatrix}, \quad H_e = \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta + D_{BH}K^c) & \frac{1}{\sqrt{N}}D_{BH}K^c_D \\ k_D K^c & k_D K^c_D \end{pmatrix}, \quad E_e = \begin{pmatrix} D_{GH} & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose that the sensors with indices in ω experience outages. The closed-loop system matrices become

$$\bar{F}_e = \begin{pmatrix} A_\delta + B_\delta K^c & B_\delta K^c_D \\ \sqrt{N}G_{\delta,C}K^d - K^o_{D,\omega}(C_{\delta,\omega} + D_{BC,\omega}K^c) & A_e - K^o_D(C_D + D_{BC,D}K^c_{C,D} - D_{BC}K^c_D) - K^o_{D,\omega}D_{BC,\omega}K^c_D \end{pmatrix},$$

$$\bar{G}_e = \begin{pmatrix} \sqrt{N}G_\delta & 0 \\ -\sqrt{N}G_{\delta,C} & \sqrt{N}K^o_{D,\bar{\omega}} \end{pmatrix}, \quad \bar{H}_e = \begin{pmatrix} \frac{1}{\sqrt{N}}(H_\delta + D_{BH}K^c) & \frac{1}{\sqrt{N}}D_{BH}K^c_D \\ k_D K^c & k_D K^c_D \end{pmatrix}, \quad \bar{E}_e = \begin{pmatrix} D_{GH} & 0 \\ 0 & 0 \end{pmatrix}.$$

The difference between the system matrices of the system with failures and the system matrices of the system with no failures is thus

$$\bar{F}_e = F_e - K^o_{\omega e} C_{\omega e}, \qquad \bar{G}_e = G_e - K^o_{\omega e}(0 \;\; \sqrt{N} I), \qquad \bar{H}_e = H_e, \qquad \bar{E}_e = E_e,$$

where

$$K^o_{\omega e} := \begin{pmatrix} 0 \\ K^o_{D,\omega} \end{pmatrix}, \qquad C_{\omega e} := (C_{\delta,\omega} + D_{BC,\omega} K^c \;\; D_{BC,\omega} K^c_D).$$

It can be shown, following the derivation in the proof of Theorem 3.5.1, that

$$\begin{aligned}
&\bar{F}_e' X_e + X_e \bar{F}_e + T \bar{F}_e' X_e \bar{F}_e + \bar{H}_e' \bar{H}_e \\
&\quad + (\bar{H}_e' \bar{E}_e + (I + T\bar{F}_e)' X_e \bar{G}_e)(\alpha^2 I - T \bar{G}_e' X_e \bar{G}_e - \bar{E}_e' \bar{E}_e)^{-1} (\bar{E}_e' \bar{H}_e + \bar{G}_e' X_e (I + T\bar{F}_e)) \\
&= F_e' X_e + X_e F_e + T F_e' X_e F_e + H_e' H_e + \tfrac{1}{N} \alpha^2 C_{\omega e}' C_{\omega e} \\
&\qquad + (H_e' E_e + (I + T F_e)' X_e G_e)(\alpha^2 I - T G_e' X_e G_e - E_e' E_e)^{-1} (E_e' H_e + G_e' X_e (I + T F_e)) \\
&\quad - \left(\sqrt{N} \tfrac{1}{\alpha} (I + T F_e)' X_e L_e^{-1} K^o_{\omega e} + \tfrac{1}{\sqrt{N}} \alpha C_{\omega e}' + \sqrt{N} \tfrac{1}{\alpha} T H_e' E_e (\alpha^2 I - E_e' E_e)^{-1} G_e' X_e L_e^{-1} K^o_{\omega e}\right) \\
&\qquad \cdot \left(I + T N \tfrac{1}{\alpha^2} K^{o\prime}_{\omega e} X_e L_e^{-1} K^o_{\omega e}\right)^{-1} \\
&\qquad \cdot \left(\sqrt{N} \tfrac{1}{\alpha} K^{o\prime}_{\omega e} X_e L_e^{-1} (I + T F_e) + \tfrac{1}{\sqrt{N}} \alpha C_{\omega e} + \sqrt{N} \tfrac{1}{\alpha} T K^{o\prime}_{\omega e} X_e L_e^{-1} G_e (\alpha^2 I - E_e' E_e)^{-1} E_e' H_e\right)
\end{aligned}$$

where

$$L_e := I - T G_e (\alpha^2 I - E_e' E_e)^{-1} G_e' X_e.$$

Thus,

$$\begin{aligned}
&\bar{F}_e' X_e + X_e \bar{F}_e + T \bar{F}_e' X_e \bar{F}_e + \bar{H}_e' \bar{H}_e \\
&\quad + (\bar{H}_e' \bar{E}_e + (I + T\bar{F}_e)' X_e \bar{G}_e)(\alpha^2 I - T \bar{G}_e' X_e \bar{G}_e - \bar{E}_e' \bar{E}_e)^{-1} (\bar{E}_e' \bar{H}_e + \bar{G}_e' X_e (I + T\bar{F}_e)) \leq 0
\end{aligned}$$

for all $\omega \subseteq \Omega$ if

$$\begin{aligned}
&F_e' X_e + X_e F_e + T F_e' X_e F_e + H_e' H_e + \tfrac{1}{N} \alpha^2 C_{\Omega e}' C_{\Omega e} \\
&\quad + (H_e' E_e + (I + T F_e)' X_e G_e)(\alpha^2 I - T G_e' X_e G_e - E_e' E_e)^{-1} (E_e' H_e + G_e' X_e (I + T F_e)) \leq 0
\end{aligned} \tag{C.1}$$

and

$$I + T N \frac{1}{\alpha^2} K^{o\prime}_{\omega e} X_e L_e^{-1} K^o_{\omega e} > 0.$$

The term $\frac{1}{N} \alpha^2 C_{\Omega e}' C_{\Omega e}$ can be expanded to

$$\begin{pmatrix} \frac{1}{N} \alpha^2 (C_{\delta,\Omega} + D_{BC,\Omega} K^c)' (C_{\delta,\Omega} + D_{BC,\Omega} K^c) & \frac{1}{N} \alpha^2 (C_{\delta,\Omega} + D_{BC,\Omega} K^c)' (D_{BC,\Omega} K^c_D) \\ \frac{1}{N} \alpha^2 (D_{BC,\Omega} K^c_D)' (C_{\delta,\Omega} + D_{BC,\Omega} K^c) & \frac{1}{N} \alpha^2 (D_{BC,\Omega} K^c_D)' (D_{BC,\Omega} K^c_D) \end{pmatrix}.$$

Let $X_e = \begin{pmatrix} X & 0 \\ 0 & X_1 \end{pmatrix}$ in (C.1). First, consider the upper-left element. It can be expanded out, and the same choice of K^d as in the derivation of the controller satisfying Problem 3.2.4 forces the last term to zero. That leaves

$$\begin{aligned} 0 &= (A_\delta + B_\delta K^c)'X + X(A_\delta + B_\delta K^c) + T(A_\delta + B_\delta K^c)'X(A_\delta + B_\delta K^c) \\ &\quad + K^{c\prime}(k_D' k_D)K^c + \tfrac{1}{N}(H_\delta + D_{BH}K^c)'(H_\delta + D_{BH}K^c) \\ &\quad + \tfrac{1}{N}\alpha^2 (C_{\delta,\Omega} + D_{BC,\Omega}K^c)'(C_{\delta,\Omega} + D_{BC,\Omega}K^c) \\ &\quad + \left(\tfrac{1}{\sqrt{N}}(H_\delta + D_{BH}K^c)'D_{GH} + \sqrt{N}(I + T(A_\delta + B_\delta K^c))'XG_\delta\right)\Upsilon^{-1} \\ &\qquad \cdot \left(\tfrac{1}{\sqrt{N}}D_{GH}'(H_\delta + D_{BH}K^c) + \sqrt{N}G_\delta' X(I + T(A_\delta + B_\delta K^c))\right). \end{aligned} \tag{C.2}$$

The upper-right element reduces to

$$\begin{aligned} 0 &= K^{c\prime}[k_D' k_D + TB_\delta' XB_\delta + \tfrac{1}{N}D_{BH}'D_{BH} \\ &\qquad + \left(\tfrac{1}{\sqrt{N}}D_{BH}'D_{GH} + \sqrt{N}TB_\delta' XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D_{GH}'D_{BH} + \sqrt{N}TG_\delta' XB_\delta\right)]K_D^c \\ &\quad + [(I + TA_\delta)'XB_\delta + \tfrac{1}{N}H_\delta' D_{BH} \\ &\qquad + \left(\tfrac{1}{\sqrt{N}}H_\delta' D_{GH} + \sqrt{N}(I + TA_\delta)'XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D_{GH}'D_{BH} + \sqrt{N}TG_\delta' XB_\delta\right)]K_D^c \\ &\quad + \tfrac{1}{N}\alpha^2(C_{\delta,\Omega} + D_{BC,\Omega}K^c)'(D_{BC,\Omega}K_D^c), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} 0 &= K^{c\prime}[k_D' k_D + TB_\delta' XB_\delta + \tfrac{1}{N}D_{BH}'D_{BH} + \tfrac{1}{N}\alpha^2 D_{BC,\Omega}'D_{BC,\Omega} \\ &\qquad + \left(\tfrac{1}{\sqrt{N}}D_{BH}'D_{GH} + \sqrt{N}TB_\delta' XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D_{GH}'D_{BH} + \sqrt{N}TG_\delta' XB_\delta\right)]K_D^c \\ &\quad + [(I + TA_\delta)'XB_\delta + \tfrac{1}{N}H_\delta' D_{BH} + \tfrac{1}{N}\alpha^2 C_{\delta,\Omega}'D_{BC,\Omega} \\ &\qquad + \left(\tfrac{1}{\sqrt{N}}H_\delta' D_{GH} + \sqrt{N}(I + TA_\delta)'XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D_{GH}'D_{BH} + \sqrt{N}TG_\delta' XB_\delta\right)]K_D^c. \end{aligned}$$

This is satisfied by (4.15). With this choice of K^c, the design equation for X, (C.2), becomes (4.16). The lower-right element becomes, for $T \neq 0$,

$$\begin{aligned} X_1 &= [I + T(A_e - G_{\delta,C}\Upsilon^{-1}[D_{GH}'D_{BH} + TNG_\delta' XB_\delta]K_D^c) \\ &\qquad\qquad - TK_D^o(C_D + D_{BC,D}K_{C,D}^c - D_{BC}K_D^c)]' \\ &\quad \cdot X_1\left(I - TNG_{\delta,C}\Upsilon^{-1}G_{\delta,C}'X_1 - TN\tfrac{1}{\alpha^2}K_D^o K_D^{o\prime}X_1\right)^{-1} \\ &\quad \cdot [I + T(A_e - G_{\delta,C}\Upsilon^{-1}[D_{GH}'D_{BH} + TNG_\delta' XB_\delta]K_D^c) \\ &\qquad\qquad - TK_D^o(C_D + D_{BC,D}K_{C,D}^c - D_{BC}K_D^c)] \\ &\quad + TK_D^{c\prime}[k_D' k_D + TB_\delta' XB_\delta + \tfrac{1}{N}D_{BH}'D_{BH} \\ &\qquad\qquad + \left(\tfrac{1}{\sqrt{N}}D_{BH}'D_{GH} + \sqrt{N}TB_\delta' XG_\delta\right)\Upsilon^{-1}\left(\tfrac{1}{\sqrt{N}}D_{GH}'D_{BH} + \sqrt{N}TG_\delta' XB_\delta\right)]K_D^c \\ &\quad + K_D^{c\prime}D_{BC,\Omega}'D_{BC,\Omega}K_D^c. \end{aligned}$$

The last term can be incorporated into the previous grouping. Then, the inversion and change of variables results in (4.17).

The derivation of conditions to guarantee condition (ii) of the generalized bounded real lemma, $I + TN\frac{1}{\alpha^2}K_{we}^{o\prime}X_eL_e{}^{-1}K_{we}^{o} > 0$, and the detectability condition, proceeds as in the proof of every other sensor-outage reliable case. □

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